Weak Computability and Representation of Real Numbers

Xizhong Zheng [b,*] Robert Rettinger [a], Romain Gengler [b],

[a] *Theoretische Informatik II, FernUniversität Hagen, 58084 Hagen, Germany*

[b] *Theoretische Informatik, BTU Cottbus, 03044 Cottbus, Germany*

Abstract

Analogous to Ershov's hierarchy for Δ_2^0-subsets of natural numbers we discuss the similar hierarchy for recursively approximable real numbers. Namely, we define the k-computability for natural number k and f-computability for function f. We will show that these notions are not equivalent for different representations of real numbers based on Cauchy sequence, Dedekind cut and binary expansion.

Key words: Computability of Real Number, Ershov's Hierarchy

1 Introduction

In classical mathematics, real numbers are represented typically by Dedekind cuts, Cauchy sequences of rational numbers and binary or decimal expansions. The effectivization of these approaches give us the equivalent definitions of computable real numbers. This notion was first explored by Alan Turing in his famous paper [13] where the Turing machine is also introduced. According to Turing, *the computable numbers may be described briefly as the real numbers whose expressions as a decimal are calculable by finite means* (page 230, [13]). In other words, a real number $x \in [0;1]$ [1] is called *computable* if there is a computable function $f : \mathbb{N} \to \{0, 1, \cdots, 9\}$ such that $x = \sum_{i \in \mathbb{N}} f(i) \cdot 10^{-i}$. Robinson [8], and independently also Myhill [6], has observed that computable real numbers can be equivalently defined via Dedekind cuts, binary expansions or Cauchy sequences of rational numbers.

* Corresponding author. email: zheng@informatik.tu-cottbus.de

[1] In this paper we restrict ourselves to the unit interval $[0;1]$. For any $n \in \mathbb{N}$, the real numbers $y := x \pm n$ and x are regarded as being of the same computability.

Theorem 1.1 (Robinson [8], Myhill [6] and Rice [7]) *Let $x \in [0;1]$ be a real number. Then the following are equivalent.*

(1) x is computable;
(2) The Dedekind cut $L_x := \{r \in \mathbb{Q} : r < x\}$ of x is a recursive set;
(3) There is a recursive set $A \subseteq \mathbb{N}$ such that $x = x_A := \sum_{i \in A} 2^{-i}$;
(4) There is a computable sequence (x_s) of rational numbers which converges to x effectively in the sense that

$$(\forall s, t \in \mathbb{N}) \left(t \geq s \Longrightarrow |x_s - x_t| \leq 2^{-s} \right). \tag{1}$$

Obviously, not every real number is computable because there are countable many computable real numbers but class of real numbers is not countable. Specker [12] can even construct a non-computable real number by defining a computable sequence (x_s) of rational numbers by $x_s := x_{A_s}$, where (A_s) is an effective enumeration of a non-computable r.e. set $A \subseteq \mathbb{N}$. This means that the extra condition (1) of the effective convergence is essential for the computability of x. Besides, the condition (1) of effective convergence can obviously replaced by

$$(\forall s \in \mathbb{N})(|x - x_s| \leq 2^{-s}) \tag{2}$$

or even by

$$(\forall s, t \in \mathbb{N}) \left(t \geq m(s) \Longrightarrow |x - x_t| \leq 2^{-s} \right) \tag{3}$$

for some computable function $m : \mathbb{N} \to \mathbb{N}$. In this case, the function m is often called a *modulus of convergence* of the sequence (x_s).

As observed by Specker [12], Theorem 1.1 does not hold if the effectivization to the primitive recursive instead of computable level are considered. Let $\mathfrak{R}_1$ be the class of all limits of primitive recursive sequences of rational numbers which converge primitive recursively (i.e., have a primitive recursive modulus of convergence), $\mathfrak{R}_2$ the class of all real numbers of primitive recursive binary expansions and $\mathfrak{R}_3$ include all real numbers of primitive recursive Dedekind cuts. It is shown in [12] that $\mathfrak{R}_3 \subsetneq \mathfrak{R}_2 \subsetneq \mathfrak{R}_1$. For polynomial time computability of real numbers, Ko [5] shows this dependence on representations of real numbers too. Let $\mathcal{P}_C$ be the class of limits of all polynomial time computable sequences of dyadic rational numbers which converges effectively, $\mathcal{P}_D$ contain all real numbers of polynomial time computable Dedekind cuts and $\mathcal{P}_B$ be the class of real numbers whose binary expansions are polynomial time computable (with the input n written in unary notation). Ko [5] shows that $\mathcal{P}_D = \mathcal{P}_B \subsetneq \mathcal{P}_C$ and $\mathcal{P}_C$ is a real closed field while $\mathcal{P}_D$ is not closed under addition and subtraction. In [5], the dyadic rational numbers $\mathbb{D} := \cup_{n \in \mathbb{N}} \mathbb{D}_n$ for

$\mathbb{D}_n := \{m \cdot 2^{-n} : m \in \mathbb{N}\}$ instead of $\mathbb{Q}$ is used as base set. For the complexity discussion $\mathbb{D}$ seems more natural and easier to use. But for computability it makes no essential difference and we use both $\mathbb{D}$ and $\mathbb{Q}$ in this paper.

In this paper, we investigate similar classes where we weaken the notion of computability in several quite natural ways instead of strengthening this notion. A typical approach to explore the non-computable objects is to classify them into equivalent classes or so-called *degrees* by various reductions (see e.g. [11]). This can be easily implemented for real numbers by mapping each set $A \subseteq \mathbb{N}$ to a real number $x_A := \sum_{i \in A} 2^{-i}$ and then defining *Turing reduction* $x_A \leq_T x_B$ by $A \leq_T B$. This definition is robust as shown in [16,2]. The benefit of this approach is that the techniques and results from well developed recursion theory can be applied straightforwardly. For example, Ho [4] shows that, a real number x is Turing reducible to $\mathbf{0}'$, the degree of the halting problem K, iff there is a computable sequence of rational numbers which converges to x. This is a reprint of Shoenfield's Limit Lemma ([9]) in recursion theory which say that $A \leq_T K$ iff A is a limit of a computable sequence of subsets of natural numbers.

However, the classification of real numbers by Turing reductions seems not fine enough and it does not relate very closely to the analytical properties of real numbers. In this paper we will give another classification of real numbers which is analogous to the Ershov's hierarchy ([3]) for subsets of natural numbers. Notice that, if $A \subseteq \mathbb{N}$ is recursive, then there is an algorithm which tells us whether a natural number n belongs to A or not. In this case, corrections are not allowed. However, if we allow the algorithm change its mind for the membership of n to A from negative to positive but at most once, then the corresponding set A is an r.e. set. In other words, the algorithm may claim $n \notin A$ at some stage and correct its claim to that $n \in A$ at a later stage. In general, given a function $h : \mathbb{N} \to \mathbb{N}$, if the algorithm is allowed to change the answer to the question "$n \in A$?" at most $h(n)$ times for any $n \in \mathbb{N}$, then the corresponding set A is called h-r.e. according to Ershov [3]. Especially, for constant function $h(n) \equiv k$, the h-r.e. sets are called k-r.e. For computable function h, the h-r.e. sets are called ω-r.e. This introduces a classification of Δ^0_2 subsets of $\mathbb{N}$ (so called Ershov's Hierarchy). Obviously, we can transfer this hierarchy to real numbers via their binary expansions straightforwardly. More precisely, we call x_A h-binary computable if A is h-r.e. Similarly, after extending Ershov's Hierarchy to subset of rational numbers, we can call x h-Dedekind computable if the Dedekind cut of x is a h-r.e. set. For the Cauchy representation of real numbers a classification similar to Ershov's can be introduced too. In this case, we count the number of the "big jumps" of the sequence instead of the number of the "mind-changes". According to Theorem 1.1.4, x is computable if there is a computable sequence (x_s) of rational numbers which converges to x and the sequence (x_s) makes no big jumps in the sense of (1). However, if up to $h(n)$ (non-overlapped) "big jumps" are allowed, then x is called h-Cauchy computable. Thus, three kinds of h-computability of real numbers can be naturally introduced. In this paper, we will investigate these notions and compare them with other known weak computability of real numbers discussed in [14]. In this case we will find that Cauchy computability is the most natural notion, although several interesting results about binary and Dedekind computability are obtained in this paper.

2 Basic Definitions

In this section, we recall first some notions of weak computability of real numbers and Ershov's hierarchy. Then we give the precise definition of binary, Dedekind and Cauchy computability.

As mentioned in the last section, a real number x is *computable* if there is a computable sequence (x_s) of rational numbers which converges to x effectively in the sense of (1). The limit of an increasing or decreasing computable sequence of rational numbers is called *left computable* or *right computable*, respectively. Left and right computable real numbers are called *semi-computable*. If x is a difference of two left computable real numbers, then x is called *weakly computable*. According to Ambos-Spies, Weihrauch and Zheng [1], x is weakly computable iff there is a computable sequence (x_s) of rational numbers which converges to x *weakly effectively*, in the sense that $\sum_{s\in\mathbb{N}} |x_s - x_{s+1}| \leq c$ for a constant c. More generally, if x is simply the limit of a computable sequence of rational numbers, then x is called *recursively approximable*. The classes of computable, left computable, right computable, semi-computable, weakly computable and recursively approximable real numbers are denoted by $\mathbf{EC}, \mathbf{LC}, \mathbf{RC}, \mathbf{SC}, \mathbf{WC}$ and $\mathbf{RA}$, respectively.

For any finite set $A := \{x_1 < x_2 < \cdots < x_k\}$ of natural numbers, the natural number $i := 2^{x_1} + 2^{x_2} + \cdots + 2^{x_k}$ is called the *canonical index* of A. The set with canonical index i is denoted by D_i. A sequence (A_s) of finite subsets of $\mathbb{N}$ is called computable if there is a computable function $g : \mathbb{N} \to \mathbb{N}$ such that $A_s = D_{g(s)}$ for any $s \in \mathbb{N}$. Similarly, we can introduce the canonical index for subsets of dyadic rational numbers. Let $\sigma : \mathbb{N} \to \mathbb{D}$ be a one-to-one coding of the dyadic numbers. For any finite set $A \subseteq \mathbb{D}$, its *canonical index* is defined as the canonical index of the set $A_\sigma := \sigma^{-1}(A) := \{n \in \mathbb{N} : \sigma(n) \in A\}$. In this paper, the subset $A \subseteq \mathbb{D}$ of canonical index n is denoted by V_n. A sequence (A_s) of finite subsets of dyadic numbers is called *computable* if there is a computable function h such that $A_s = V_{h(s)}$ for all $s \in \mathbb{N}$.

Definition 2.1 (Ershov [3]) For any function $h : \mathbb{N} \to \mathbb{N}$, a set $A \subseteq \mathbb{N}$ is called *h-recursively enumerable* (h-r.e. for short) if there is a computable sequence (A_s) of finite subsets $A_s \subseteq \mathbb{N}$ such that

(1) $A_0 = \emptyset$ and $A = \bigcup_{i=0}^{\infty} \bigcap_{j=i}^{\infty} A_j$.
(2) $(\forall n \in \mathbb{N})(|\{s : n \in A_s \Delta A_{s+1}\}| \leq h(n))$, where $A \Delta B := (A \setminus B) \cup (B \setminus A)$ is the symmetrical difference of A and B.

In this case, the sequence (A_s) is called an *effective h-enumeration* of A. For $k \in \mathbb{N}$, a set A is called k-r.e. if it is h-r.e. for the constant function $h(n) \equiv k$ and A is ω-r.e. if it is h-r.e. for some computable function h. For the convenience, recursive sets are called 0-r.e.

Theorem 2.2 (Hierarchy Theorem, Ershov [3]) *For computable functions $f, g : \mathbb{N} \to \mathbb{N}$, if $(\exists^\infty n \in \mathbb{N})(f(n) < g(n))$, then there is a g-r.e. set which is not f-r.e. Thus, there is an ω-r.e. set which is not k-r.e. for any $k \in \mathbb{N}$; there is a $(k+1)$-r.e. set which is not k-r.e. (for every $k \in \mathbb{N}$), and there is also a Δ_2^0-set which is not ω-r.e.*

The definition of h-r.e., k-r.e. and ω-r.e. subsets of natural numbers can be transferred straightforwardly to subsets of dyadic rational numbers. Of course, h should be a function of type $h : \mathbb{D} \to \mathbb{N}$ in this case. This should be clear from context and is usually not indicated expressively later. Thus, we can easily introduce corresponding hierarchy for real numbers by means of binary or Dedekind representation of real numbers. However, if the real numbers are represented by the sequences of rational numbers, we should count the number of their jumps of of certain size. More precisely, we have the following definition.

Definition 2.3 Let (x_s) be a sequence of real numbers which converges to x and $n \in \mathbb{N}$.

(1) An *n-jump* of the sequence (x_s) is an index pair (i, j) such that $n < i < j$ and $2^{-n} \leq |x_i - x_j| < 2^{-n+1}$.
(2) The *n-divergence* of (x_s) is the maximal number of non-nested n-jump pairs of (x_s), i.e., the maximal natural number m such that there is a chain $n < i_1 < j_1 \leq i_2 < j_2 \leq \cdots \leq i_m < j_m$ with $2^{-n} \leq |x_{i_t} - x_{j_t}| < 2^{-n+1}$ for $t = 1, 2, \cdots, m$.
(3) For $h : \mathbb{N} \to \mathbb{N}$, if the n-divergence of (x_s) is bounded by $h(n)$ for any $n \in \mathbb{N}$, then we say that (x_s) *converges to x h-effectively.*

Definition 2.4 Let $x \in [0; 1]$ be a real number and $h : \mathbb{N} \to \mathbb{N}$ a function.

(1) x is *h-binary computable* (h-bEC for short) if there is a h-r.e. set $A \subseteq \mathbb{N}$ such that $x = x_A$;
(2) x is *h-Cauchy computable* (h-cEC for short) if there is a computable sequence (x_s) of rational numbers which converges to x h-effectively;
(3) x is *h-Dedekind computable* (h-dEC for short) if the left Dedekind $L_x := \{r \in \mathbb{Q} : r < x\}$ is a h-r.e. set.
(4) For $\delta \in \{b, c, d\}$, x is called k-δEC if x is h-δEC for the constant function

$h(n) \equiv k$. x is called ω-δEC if it is h-δEC for a computable function h.

The classes of all k-δEC, ω-δEC and h-δEC real numbers are denoted by k-$\delta\,\mathbf{EC}$, ω-$\delta\,\mathbf{EC}$ and h-$\delta\,\mathbf{EC}$, respectively, for $\delta \in \{b, c, d\}$. Besides, we denote also $*$-$\delta\,\mathbf{EC} := \bigcup_{n\in\mathbb{N}} n$-$\delta\,\mathbf{EC}$. The following proposition follows directly from the definition.

Proposition 2.5 *For $\delta \in \{b, c, d\}$ and $f, g : \mathbb{N} \to \mathbb{N}$, the following hold.*

(1) 0-$\delta\,\mathbf{EC} = \mathbf{EC}$.
(2) k-$\delta\,\mathbf{EC} \subseteq (k+1)$-$\delta\,\mathbf{EC} \subseteq *$-$\delta\,\mathbf{EC} \subseteq \omega$-$\delta\,\mathbf{EC}$, *for any* $k \in \mathbb{N}$.
(3) If $f(n) \le g(n)$ holds for almost all $n \in \mathbb{N}$, then f-$\delta\,\mathbf{EC} \subseteq g$-$\delta\,\mathbf{EC}$.

3 Binary Computability

In this section we discuss the binary computability. From Theorem 2.2, it follows immediately that, if f, g are computable functions such that $f(n) < g(n)$ for infinitely many n, then g-$\mathbf{EC} \setminus f$-$\mathbf{EC} \neq \emptyset$. Thus, we have the following hierarchy theorem for binary computability.

Proposition 3.1 k-b$\mathbf{EC} \subsetneq (k+1)$-b$\mathbf{EC} \subsetneq *$-b$\mathbf{EC} \subsetneq \omega$-b$\mathbf{EC}$*, for any* $k \in \mathbb{N}$.

Now we compare the binary computability with semi-computability. It turns out that $\mathbf{SC}$ is incomparable with $*$-b$\mathbf{EC}$ but included properly in ω-b$\mathbf{EC}$.

Theorem 3.2 *(1)* $\mathbf{SC} \subsetneq \omega$-b$\mathbf{EC}$.
(2) $\mathbf{SC} \not\subseteq *$-b$\mathbf{EC}$.
(3) 2-b$\mathbf{EC} \not\subseteq \mathbf{SC}$.

Proof. (1). As it is pointed out by Soare ([10], page 217), that if the real number x_A is semi-computable, then set A is 2^{n+1}-r.e. By Theorem 2.2, there is an ω-r.e. set A which is not 2^{n+1}-r.e. Therefore, the real number x_A is ω-bEC but not semi-computable. That is, $\mathbf{SC} \subsetneq \omega$-b$\mathbf{EC}$.

(2). We construct in stages a set $A \subseteq \mathbb{N}$ such that x_A is left computable but it is not $*$-bEC. To this end, set A has to satisfy, for all $i, j \in \mathbb{N}$, the following requirements.

$$R_{\langle i,j\rangle} : \ (D_{\varphi_i(s)})_s \text{ is an effective } j\text{-enumeration} \Longrightarrow A \neq \lim_{s\to\infty} D_{\varphi_i(s)}.$$

where (φ_i) is an effective enumeration of all computable partial functions $\varphi :\subseteq \mathbb{N} \to \mathbb{N}$. To satisfy R_e for $e := \langle i, j\rangle$, we choose an $n_e > j$. We put n_e into A as long as n_e is not in $D_{\varphi_i(s)}$. If n_e enters $D_{\varphi_i(s)}$ for some s, then we take n_e

out of A. n_e may be put into A again if n_e leaves $D_{\varphi_i(t)}$ for some $t > s$, and so on. Obviously, we need only to change the membership of n_e to A at most j times and the strategy succeeds eventually. To make x_A left computable, we reserve an interval $[m_e; n_e]$ of natural numbers with $n_e - m_e > j$ exclusively for R_e and put a new element from this interval into A whenever n_e is taken out of A.

(3). In [1], Ambos-Spies, Weihrauch and Zheng (Theorem 4.8 of [1]) have shown that, if two r.e. sets $A, B \subseteq \mathbb{N}$ are Turing incomparable, i.e., $A \not\leq_T B$ & $B \not\leq_T A$, then the real number $x_{A\oplus\overline{B}}$ is not semi-computable, where $\overline{B}$ is the complement of B and $A \oplus \overline{B} := \{2n : n \in A\} \cup \{2n+1 : n \in \overline{B}\}$ is the join of the sets A and $\overline{B}$. On the other hand, for any r.e. sets A, B, the join $A \oplus \overline{B} := (2A \cup (2\mathbb{N}+1)) \setminus (2B+1)$ is a 2-r.e. set and hence $x_{A\oplus\overline{B}}$ is 2-bEC. $\square$

From the items (2) and (3) of the theorem 3.2, the class $*$-b**EC** is not comparable with the class **SC**. For weakly computable real numbers, we have at first the following result.

Lemma 3.3 *For any natural number n, we have n-b***EC** $\subseteq$ **WC**. *Therefore,* $*$-b**EC** $\subsetneq$ **WC**.

Proof. The inclusion part can be proved by an induction on n. For $n = 0$ it holds trivially. Assume by induction hypothesis that x_C is weakly computable for any n-r.e. set $C \subseteq \mathbb{N}$. If A is an $(n+1)$-r.e. set, then there are r.e. set A_1 and n-r.e. set A_2 such that $A = A_1 - A_2$. This implies that $x_A = x_{A_1\cup A_2} - x_{A_2}$. The set $A_1 \cup A_2$ is obviously also n-r.e, and hence both the real numbers $x_{A_1\cup A_2}$ and x_{A_2} are weakly computable. Since **WC** is closed under the arithmetical operation, x_A is weakly computable too.

The inequality part follows directly from the item (3) of the theorem 3.2. $\square$

Theorem 3.4 **WC** $\not\subseteq$ ω-b**EC** *and* ω-b**EC** $\not\subseteq$ **WC**

Proof. In [15] Zheng shows that there are r.e. sets $A, B \subseteq \mathbb{N}$ such that the set $C \subseteq \mathbb{N}$ defined by $x_C := x_A - x_B$ is not of ω-r.e. Turing degree. This means that x_C is weakly computable but not ω-bEC. That is, **WC** $\not\subseteq$ ω-b**EC**.

The part ω-b**EC** $\not\subseteq$ **WC** follows immediately from a result of [1], that if $x_{A\oplus\emptyset}$ is weakly computable, then A is a 2^{3n}-r.e. set. By Ershov's Hierarchy Theorem 2.2, we can choose an ω-r.e. A which is not 2^{3n}-r.e. Then $B := A \oplus \emptyset$ is obviously also an ω-r.e. set and hence x_B is ω-bEC. But x_B is not weakly computable because A is not 2^{3n}-r.e. $\square$

4 Dedekind Computability

We investigate Dedekind computability in this section. Different from the case of binary computability, the hierarchy theorem does not hold now. However the class of all ω-Dedekind computable real numbers is incomparable with weakly computable real numbers too. Between ω-binary and ω-Dedekind computability we have the following result at first.

Theorem 4.1 ω-b$\mathbf{EC} \subseteq \omega$-d$\mathbf{EC}$

Proof. We consider only the real numbers from the unit interval $[0;1]$. Suppose that $x_A \in \omega$-b$\mathbf{EC}$, i.e., A is an ω-r.e. set. Then there is a computable function h and an effective h-enumeration (A_s) such that $\lim_{s\to\infty} A_s = A$. We define a computable sequence (E_s) of finite subsets of dyadic numbers by $E_s := \{r \in \mathbb{D}_s : r < x_{A_s}\}$, where $\mathbb{D}_s$ is the set of all dyadic rational numbers of precision s. It is easy to see that the limit $E := \lim_s E_s$ exists and it is in fact the left Dedekind cut of the real number x_A. For any dyadic rational number r, denote by $d(r)$ its precision, namely $d(r) := \min\{s \in \mathbb{N} : r \in \mathbb{D}_s\}$. Then, $d : \mathbb{D} \to \mathbb{N}$ is a computable function such that, for any $r \in \mathbb{D}$, there is a finite set $B_r \subseteq \{0, 1, \cdots, d(r)\}$ with $r = x_{B_r}$. For any $r \in \mathbb{D}$ and $s \in \mathbb{N}$, if $r \in E_s \Delta E_{s+1}$, namely, $x_{A_s} < r$ & $r \leq x_{A_{s+1}}$ or $r \leq x_{A_s}$ & $x_{A_{s+1}} < r$, then there must be a natural number $n \leq d(r)$ such that $n \in A_s \Delta A_{s+1}$. This means that the sequence (E_s) is a g-effective enumeration of E for the computable function $g : \mathbb{D} \to \mathbb{N}$ defined by $g(r) := \sum_{i \leq d(r)} h(i)$. Thus, x is a g-dEC and hence an ω-dEC real number. $\square$

The next result shows that the class $*$-d$\mathbf{EC}$ collapses to $\mathbf{SC}$ and hence the hierarchy theorem does not hold.

Lemma 4.2 *(1)* 1-d$\mathbf{EC} = \mathbf{LC}$ *and* $\mathbf{SC} \subseteq$ 2-d$\mathbf{EC}$.
(2) $*$-d$\mathbf{EC} = \mathbf{SC}$.

Proof. 1. This follows directly from the definition.

2. By item 1, it suffices to prove that $*$-d$\mathbf{EC} \subseteq \mathbf{SC}$. For any $x \in *$-d$\mathbf{EC}$, let $k := \min\{n : x \in n\text{-d}\mathbf{EC}\}$. Then the Dedekind cut L_x of x is a k-r.e. but not $(k-1)$-r.e. set. Let (A_s) be an effective k-enumeration of L_x. Then there are infinitely many $r \in \mathbb{D}$ such that $|\{s \in \mathbb{N} : r \in A_{s+1} \backslash A_s\}| = k$. Let $O_k := \{r \in \mathbb{D} : |\{s \in \mathbb{N} : r \in A_{s+1} \backslash A_s\}| = k\}$. Obviously, O_k is an r.e. set. If $k > 0$ and k is even, then $x < r$ for any $r \in O_k$ and we can choose a decreasing computable sequence (r_s) from O_k such that $\lim r_s = x$. Otherwise, there is a rational number y such that $x < y < r$ for any $r \in O_k$. In this case, we can construct an effective $(k-1)$-enumeration of L_x by allowing any $r > y$ enters L_x at most $k/2 - 1$ times. This contradicts the hypothesis. Thus x is a right

computable real number.

Similarly, if k is odd, then x is left computable. $\square$

Our next theorem shows that not every weakly computable real number is ω-Dedekind computable.

Theorem 4.3 **WC** $\not\subseteq$ ω-d**EC**.

Proof. We will construct two recursive enumerations (A_s) and (B_s) of the r.e. sets A and B respectively in stages. At any stage s, the finite set C_s is defined simply by $x_{C_s} = x_{A_s} - x_{B_s}$. Let $C := \lim_{s\to\infty} C_s = \bigcup_{s\in\mathbb{N}} \bigcap_{t\geq s} C_t$. Then x_C is a weakly computable real number.

In order to guarantee that x_C is not ω-dEC, we have to diagonalize against the class ω-d**EC**. Let (φ_i) and (ψ_j) be effective enumerations of all partial computable functions $\varphi_i :\subseteq \mathbb{N} \to \mathbb{N}$ and $\psi_j :\subseteq \mathbb{D} \to \mathbb{N}$, respectively. If y is an ω-dEC number, i.e., the left Dedekind cut L_y of y is an ω-r.e. set, then there are a computable sequence (E_s) of finite subsets of dyadic numbers and a total computable function $\psi : \mathbb{D} \to \mathbb{N}$ such that (E_s) is a ψ-enumeration and $L_y = E := \lim_{s\to\infty} E_s$. For the computable sequence (E_s), there is a computable function φ such that $E_s = V_{\varphi(s)}$ for all $s \in \mathbb{N}$. Thus, it suffices to satisfy, for all $i, j \in \mathbb{N}$, the following requirements:

$$R_{\langle i,j\rangle}: \quad \left.\begin{array}{l} \varphi_i \text{ and } \psi_j \text{ are total functions} \\ (V_{\varphi_i(s)})_{s\in\mathbb{N}} \text{ is a } \psi_j\text{-enumeration, and} \\ E_i := \lim_s V_{\varphi_i(s)} \text{ is a Dedekind cut} \end{array}\right\} \Longrightarrow \sup(E_i) \neq x_C.$$

The strategy to satisfy a single requirement R_e for $e = \langle i, j\rangle$ looks like the following. Suppose that the presumptions of R_e hold. Given arbitrary three finite sets $A, B, C \subseteq \mathbb{N}$ such that $x_C = x_A - x_B$. Choose a natural number n_e such that $n_e > \max(A \cup B \cup C)$ and define $m_e := n_e + \psi_j(y_e) + 2$ where $y_e := x_{C\cup\{n_e\}}$ is a dyadic rational number. Then, we put the numbers n_e and m_e into A but they will never be put into B in the construction (R_e enters the phase 1). Thus, n_e and m_e belong also to C now and m_e will remain in C forever. The natural numbers between n_e and m_e are reserved exclusively for the requirement R_e and no natural numbers less than n_e are allowed to enter or leave the sets A, B and C. At this case, we have $y_e < x_C$. If there is no stage s such that $y_e \in V_{\varphi_i(s)}$, then $y_e \notin E_i := \lim_s V_{\varphi_i(s)}$. This means that $\sup(E_i) \leq y_e$ if E_i is a Dedekind cut. Therefore, $x_C > y_e \geq \sup(E_i)$ and hence the requirement R_e is satisfied.

Otherwise, suppose that, at some stage s_0, the diadic rational number y_e enters the set $V_{\varphi_i(t_0)}$ for some $t_0 \leq s_0$, i.e., the inequality $\sup(E_i) > y_e$ seems correct at the moment, then we put the natural number $n_e + 1$ into B (but not into A) (R_e in phase 2). This forces the natural number n_e to leave the set C because of the condition $x_C = x_A - x_B$ and implies at the same time that $x_C < y_e$. In other words, the inequality $x_C < y_e < \sup(E_i)$ seems correct at the moment. If at a later stage $s_1 > s_0$, the number y_e leaves the set $V_{\varphi_i(t_1)}$ for some $t_1 > t_0$, i.e., the inequality $\sup(E_i) > y_e$ seems wrong again, then we put the natural number $n_e + 1$ into A and this forces the number n_e enters the set C again (R_e in phase 1 again). In this case, the inequality $\sup(E_i) \leq y_e < x_C$ becomes correct. Similarly, if at stage $s_2 > s_1$, y_e enters the set $V_{\varphi_i(t_2)}$ for some $t_2 > t_1$, then we can force n_e leave C by putting $n_e + 2$ into B, and so on. Since $(V_{\varphi_i(s)})_{s\in\mathbb{N}}$ is a φ_i-enumeration, this can happen at most $\varphi_i(y_e)$ times. Therefore, the natural numbers reserved in the interval $(n_e; m_e)$ suffice to satisfy the requirement R_e eventually and this strategy successes.

To satisfy all the requirements R_e simultaneously, we apply the finite injury priority method. Here R_i has a higher priority than R_j means that $i < j$. More precisely, we have the following formal construction of the recursive enumerations (A_s) and (B_s).

Stage $s = 0$: Let $A_0 = B_0 = C_0 := \emptyset$, $n_{e,0} := 4e$, $z_{e,0} := 4e$, $m_{e,0} := 4e + 3$ and $t_{e,0} := -1$ for all $e \in \mathbb{N}$. All requirements R_e are in the phase 0.

Stage $s+1$: Given A_s, B_s, C_s, $n_{e,s}$, $z_{e,s}$, $m_{e,s}$ and $t_{e,s}$ for any natural number e. Let $y_{e,s} := x_{(C_s \upharpoonright n_{e,s}) \cup \{n_{e,s}\}}$. A requirement R_e, ($e := \langle i, j \rangle$) *requires attention* if $e \leq s$ and one of the following cases appear:

Case 1: The requirement R_e is in the phase 0 and $\psi_{j,s}(y_{e,s})$ is defined,

Case 2: The requirementR_e is in the phase 1 and there is a natural number t with $t_{e,s} < t \leq s$ such that $y_{e,s} \in V_{\varphi_{i,s}(t)}$; or

Case 3: The requirement R_e is in the phase 2, $z_{e,s} + 1 < m_{e,s}$ and there is a natural number t with $t_{e,s} < t \leq s$ such that $y_{e,s} \notin V_{\varphi_{i,s}(t)}$.

If no requirement R_e requires attention, then go directly to the next stage. Otherwise, let R_e be the requirement of highest priority which requires attention. We will do according to following cases:

Case 1: If the requirement R_e is in the phase 0, then define $m_{e,s+1} := \psi_{j,s}(y_{e,s}) + 2$ and $z_{e,s+1} := n_{e,s} + 1$, put $n_{e,s}$ and $m_{e,s+1}$ into A, i.e., $A_{s+1} := A_s \cup \{n_{e,s}, m_{e,s+1}\}$, and at the last put R_e into the phase 1. In this case, we say that the requirement $R_{e'}$ is injured (by R_e) if $e' > e$ and $R_{e'}$ is in the phase 1 or 2 at the stage s. All requirements $R_{e'}$ for $e' > e$ are initialized by putting $R_{e'}$ into phase 0 and defining the new parameters by

$$\begin{cases} n_{e',s+1} & := m_{e,s+1} + 4e' \\ m_{e',s+1} & := m_{e,s+1} + 4e' + 3 \\ z_{e',s+1} & := m_{e,s+1} + 4e' + 1 \\ t_{e',s+1} & := -1. \end{cases}$$

Case 2: If the requirement R_e in the phase 1, then simply put $z_{e,s}$ into B, i.e., define $B_{s+1} := B_s \cup \{z_{e,s}\}$. The requirement R_e is set into the phase 2.

Case 3: If the requirement R_e is in the phase 2, then put $z_{e,s}$ into A, i.e., define $A_{s+1} = A_s \cup \{z_{e,s}\}$. Furthermore, we define $z_{e,s+1} := z_{e,s} + 1$ and change the phase of R_e into phase 1.

In all three cases, we say that the requirement R_e *receives attention*. All the paraments which is not explicitly redefined above remain the same as in the stage s. This ends the formal construction.

Obviously, the sequences (A_s) and (B_s) are recursive enumerations of sets $A := \cup_s A_s$ and $B := \cup_s B_s$, respectively. That is, A, B are r.e. sets. And hence $x_C := x_A - x_B$ is a weakly computable real numbers. For any e, the requirement R_e can only be injured by some requirement R_i of higher priority, i.e., $i < e$. Without injury, R_e can receive attention at most $2\psi_j(y_e) + 1$ times, if $\psi_j(y_e)$ is defined or does not require attention at all otherwise. Therefore, by an induction, we can show that all requirements R_e receive attention only finitely often.

Furthermore, we can show that the real number x_C satisfies all requirements indeed. Suppose that the presumptions of the requirement R_e hold and R_e does not require attention after stage s_0 any more for some $s_0 \in \mathbb{N}$. Since ψ_j is a total function, R_e is not in the phase 0 after stage s_0, otherwise R_e requires attention according to case 1. If R_e is in the phase 1, then $y_e := y_{e,s_0}$ will not enter the sets $V_{\varphi_i(t)}$ for any $t \geq t_{e,s_0}$ and hence $y_e \notin E_i := \lim_s V_{\varphi_i(t)}$. This means that $\sup(E_i) \leq y_e$, because E_i is a left Dedekind cut. On the other hand, by the construction, we have $C \upharpoonright (n_e + 1) = (C_{s_0} \upharpoonright n_e) \cup \{n_e\}$ for $n_e := n_{e,s_0}$ and $m_{e,s_0} \in C$. Therefore, we have $y_e = x_{(C_{s_0} \upharpoonright n_e) \cup \{n_e\}} = x_{C \upharpoonright (n_e+1)} < x_{C \upharpoonright (n_e+1)} + 2^{-m_{e,s_0}} < x_C$. This implies that $\sup(E_i) < x_C$ and R_e is satisfied.

Otherwise, if R_e is in the phase 2 after stage s_0, then y_e should remain in $V_{\varphi_i(t)}$ for any $t \geq t_{e,s_0}$ and hence $y_e \in E_i := \lim_s V_{\varphi_i(t)}$. This means that $y_e \leq \sup(E_i)$. On the other hand, R_e remains in the phase 2 after stage s_0 means that $z_{e,s_0} \in B$ but $z_{e,s_0} \notin A$. This implies that $n_e \notin C_{s_0}$ and hence $n_e \notin C$ because we have $C \upharpoonright m_e = (C_{s_0} \upharpoonright m_e)$ by the construction. This implies that $x_C < x_{(C_{s_0} \upharpoonright n_e) \cup \{n_e\}} = y_e \leq \sup(E_i)$. That is, the requirement R_e

is satisfied in this case too. This concludes that our construction succeeds. □

Corollary 4.4 *The class* ω-d**EC** *is incomparable with the class* **WC** *and hence the class* $*$-d**EC** *is a proper subset of* ω-d**EC**.

Proof. By Theorem 4.3, we have the noninclusion $\mathbf{WC} \not\subseteq \omega\text{-d}\mathbf{EC}$. Another noninclusion $\omega\text{-d}\mathbf{EC} \not\subseteq \mathbf{WC}$ follows from directly the results $\omega\text{-b}\mathbf{EC} \not\subseteq \mathbf{WC}$ of Theorem 3.4 and $\omega\text{-b}\mathbf{EC} \subseteq \omega\text{-d}\mathbf{EC}$ of Theorem 4.1. □

Corollary 4.5 *The class* ω-d**EC** *is not closed under the operations of addition and subtraction.*

Proof. By Lemma 4.2.2, we have $\mathbf{SC} \subseteq \omega\text{-d}\mathbf{EC}$. If ω-d**EC** is closed under addition and subtraction, then $\mathbf{WC} \subseteq \omega\text{-d}\mathbf{EC}$ holds because **WC** is the closure of **SC** under addition and substraction. This contradicts to Theorem 4.3. □

5 Cauchy Computability

We discuss the Cauchy computability in this section. We will show that, all classes k-c**EC** (for any constant $k \in \mathbb{N}$) and $*$-c**EC** are incomparable with the classes **LC** and **SC**,. The class $*$-c**EC** is not closed under the addition. However the hierarchy theorem holds. At last we show that the class ω-c**EC** is in fact the image class of all left computable real numbers under total computable real functions.

Theorem 5.1 *(1) The class* ω-c**EC** *is a field;*
(2) $1\text{-c}\mathbf{EC} \not\subseteq \mathbf{SC} \not\subseteq *\text{-c}\mathbf{EC}$*;*
(3) $*\text{-c}\mathbf{EC} \subsetneq \mathbf{WC} \subsetneq \omega\text{-c}\mathbf{EC} \subsetneq \mathbf{RA}$.

Proof. 1. Let $x, y \in \omega$-c**EC**, (x_s) and (y_s) be computable sequences of rational numbers which converge to x and y d_1-effectively and d_2-effectively, respectively, where d_1 and d_2 are computable functions.

For any $n, i, j \in \mathbb{N}$ with $i < j$, if $|x_i - x_j| < 2^{-n}$ and $|y_i - y_j| < 2^{-n}$, then $|(x_i + y_i) - (x_j + y_j)| \leq |x_i - x_j| + |y_i - y_j| \leq 2^{-(n-1)}$. This implies that the n-divergence of the sequence $(x_s + y_s)$ is bounded by $d_1(n+1) + d_2(n+1)$. That is, the computable sequence $(x_s + y_s)$ converges to $x + y$ d-effectively, where $d(n) := d_1(n+1) + d_2(n+1)$. Thus, $x + y \in \omega$-c**EC** holds. Similarly we can show that $x - y, x \cdot y \in \omega$-c**EC** and (if $y \neq 0$) $x/y \in \omega$-c**EC** too.

2. "$1\text{-c}\mathbf{EC} \not\subseteq \mathbf{SC}$": Let $A, B \subseteq \mathbb{N}$ be two Turing incomparable r.e. sets and (A_s) and (B_s) the effective enumerations of A and B, respectively, such that

$$A_0 = \emptyset \,\&\, (\forall s \in \mathbb{N})(A_{2s} = A_{2s+1} \;\&\; |A_{2s+2} \setminus A_{2s+1}| = 1)$$
$$B_0 = \emptyset \,\&\, (\forall s \in \mathbb{N})(B_{2s+1} = B_{2s+2} \;\&\; |B_{2s+1} \setminus B_{2s}| = 1).$$

Then $x_{A\oplus\overline{B}} \notin \mathbf{SC}$ by Theorem 4.8 of [1] and $|(A_{s+1} \oplus \overline{B}_{s+1})\Delta(A_s \oplus \overline{B}_s)| = 1$ for any $s \in \mathbb{N}$. The sequence (x_s) defined by $x_s := x_{A_s\oplus\overline{B}_s}$ is a computable sequence of rational numbers which converges to $x_{A\oplus\overline{B}}$ 1-effectively. That is, $x_{A\oplus\overline{B}} \in 1\text{-c}\mathbf{EC} \setminus \mathbf{SC}$.

"$\mathbf{SC} \not\subseteq *\text{-c}\mathbf{EC}$": We construct an increasing computable sequence (x_s) of rational numbers such that the limit $x := \lim_s x_s$ is not k-cEC for any $k \in \mathbb{N}$. That is, x satisfies, for all $i, j \in \mathbb{N}$, the following requirements

$$R_{\langle i,j\rangle}: \quad (\varphi_i(s))_s \text{ converges } j\text{-effectively} \Longrightarrow x \neq \lim_{s\to\infty} \varphi_i(s),$$

where (φ_i) is an effective enumeration of all computable partial functions $\varphi_i :\subseteq \mathbb{N} \to \mathbb{Q}$. To satisfy the requirement R_e for $e = \langle i, j\rangle$, we choose a rational interval $[a; b]$. Let n be the minimal natural number such that $2^n \geq 3(j+1)/(b-a)$. Define $a_i := a + i \cdot 2^{-n}$ for $i \leq 3(j+1)$ and $a_{3(j+1)+1} = b$. Then the intervals $I_i := [a_i; a_{i+1}]$ have the length 2^{-n} for any $i < 3(j+1)$. We define x_0 as the middle point of I_1. If $\varphi_i(s)$ enters I_1, then define x_s as the middle point of I_4. If there is a $t > s$ such that $\varphi_i(t) \in I_4$, then define x_t as middle point of I_7, and so on. In general, if $x_{s_1} \in I_{3k+1}$ and $\varphi_i(s_2) \in I_{3k+1}$ for some $s_2 > s_1$, then redefine x_{s_2} as the middle point of I_{3k+4}. If $(\varphi_i(s))_s$ converges j-effectively, then we can always find a correct x which different from the limit $\lim_s \varphi_i(s)$, because $\varphi_i(s_1) \in I_{3k+1}$ and $\varphi_i(s_2) \in I_{3k+4}$ implies that $2^{-n+1} \leq |\varphi_i(s_1) - \varphi_i(s_2)| \leq 2^{-n+2}$.

To satisfy all requirements, it suffices to apply the above strategy to an interval tree and use the finite injury priority construction.

3. "$*\text{-c}\mathbf{EC} \subsetneq \mathbf{WC}$": Let (x_s) be a computable sequence of rational numbers which converges k-effectively to a $*$-cEC real number x for some $k \in \mathbb{N}$. For any $n \in \mathbb{N}$, let $S_n := \{s \in \mathbb{N} : 2^{-n} \leq |x_s - x_{s+1}| < 2^{-n+1}\}$. Then $\sum_{s\in\mathbb{N}} |x_s - x_{s+1}| = \sum_{n\in\mathbb{N}} \left(\sum_{s\in S_n \& s\leq n} |x_s - x_{s+1}| + \sum_{s\in S_n \& s>n} |x_s - x_{s+1}|\right) \leq \sum_{n\in\mathbb{N}}(n \cdot 2^{-n+1} + k \cdot 2^{-n+1}) \leq 8 + 2k$. That is, x is a weakly computable real number. Therefore, $*\text{-}\mathbf{EC} \subseteq \mathbf{WC}$. By the assertion $\mathbf{SC} \not\subseteq *\text{-c}\mathbf{EC}$ of the item (2), this inclusion is also proper.

"$\mathbf{WC} \subsetneq \omega\text{-c}\mathbf{EC}$": For the inclusion part is suffices to show that $\mathbf{LC} \subseteq \omega\text{-c}\mathbf{EC}$ because $\omega\text{-c}\mathbf{EC}$ is closed under the operations "$+, -$" and $\mathbf{WC}$ is the closure of $\mathbf{LC}$ under "$+, -$". Let $x \in \mathbf{LC}$ and (x_s) be an increasing computable sequence of rational numbers which converges to x. Suppose w.l.o.g. that $x - x_0 \leq 1$. Then there are at most 2^n non-overlapped pairs (i, j) of natural numbers such that $i < j$ and $x_j - x_i \geq 2^{-n}$. That is, (x_s) converges to x h-effectively for $h(n) := 2^n$. Thus x is ω-cEC and hence $\mathbf{LC} \subseteq \omega\text{-c}\mathbf{EC}$. The inequality part follows from that fact ω-b**EC** $\subseteq$ ω-c**EC** and the result ω-b**EC** $\nsubseteq$ **WC** of of Theorem 3.4

"ω-c**EC** $\subsetneq$ **RA**": We construct a computable sequence (x_s) of rational numbers converging to some x which satisfies, for all $i, j \in \mathbb{N}$, the requirements:

$$R_{\langle i,j\rangle}: \qquad (\beta_j(s))_s \text{ converges } \alpha_i\text{-effectively to } y_j \Longrightarrow x \neq y_j,$$

where (α_i) and (β_j) are effectively enumerations of all partial computable functions $\alpha_i :\subseteq \mathbb{N} \to \mathbb{N}$ and $\beta_j :\subseteq \mathbb{N} \to \mathbb{Q}$, respectively.

To satisfy a single requirement R_e for $e := \langle i, j\rangle$, we fix an rational interval I of length 2^{-3e} and divide I into eight equidistant subintervals I_k for $k < 8$. Then we define x_s as the middle point of the interval I_1 as long as $\alpha_{i,s}(3e+3)$ is not defined or the sequences $(\beta_{j,s}(t))$ does not enter the interval I_1. Otherwise, if $\alpha_{i,s_0}(3e+3)$ is defined and $\beta_{j,s_0}(t_0) \in I_1$ for some stage s_0 and some $t_0 \leq s_0$, then define x_{s_0} as the middle point of I_3. Later, if at stage $s_1 > s_0$, $\beta_{j,s_1}(t_1) \in I_3$ for some $t_1 > t_0$, then redefine x_{s_1} as middle point of I_1 again and so on. This can happen only finitely often if the sequence $(\beta_j(s))_s$ converges α_i-effectively to y_j. Therefore the limit $x := \lim_{s\to\infty} x_s$ exists and $x \neq y_j$. That is R_e is satisfied.

To satisfy all requirements R_e simultaneously, a finite injury priority construction suffices. □

Now we prove the hierarchy theorem for Cauchy computability.

Theorem 5.2 *For any computable functions f, g with $\exists^\infty n(f(n) < g(n))$, there is a g-cEC real number which is not f-cEC, i.e., g-c**EC** $\setminus$ f-c**EC** $\neq \emptyset$.*

Proof. Let $f, g : \mathbb{N} \to \mathbb{N}$ be two computable functions such that $(\exists^\infty n)(f(n) < g(n))$. We are going to construct a computable sequence (x_s) of rational numbers such that (x_s) converges g-effectively to some real number x which is not f-computable. That is, the following requirements are satisfied for any $e \in \mathbb{N}$,

$N:$ (x_s) converges g-effectively to x, and
$R_e:$ If $(\varphi_e(s))_s$ converges f-effectively, then $x \neq \lim_s \varphi_e(s)$,

where (φ_e) is an effective enumeration of all computable partial functions $\varphi_e :\subseteq: \mathbb{N} \to \mathbb{Q}$. The construction is a diagonalization with finite injury.

The strategy to satisfy a single requirement R_e is quite straightforward. Let I_e be a rational interval with length 2^{-n_e} for some $n_e \in \mathbb{N}$ such that $f(n_e) < g(n_e)$. Divide it equally into four subintervals I_i, for $i < 4$, of the length

$2^{-(n_e+2)}$. Define x_s as the middle point of the interval I_1 as long as the sequence $(\varphi_e(s))_s$ does not enter the interval I_1. Otherwise, if $\varphi_e(s)$ enters into I_1 for some s, then let x_s let the middle point of I_3. Later, if $\varphi_e(t)$ enters I_3 for some $t > s$, then let x_t be the middle point of I_1 again, and so on. If $(\varphi_e(s))_s$ converges f-effectively, then (x_s) needs at most $f(n_e) + 1 \leq g(n_e))$ jumps to guarantee that $\lim x_s \neq \lim_s \varphi_e(s)$.

To satisfy all the requirements simultaneously, we will construct an increasing sequence (n_e) of natural numbers such that $f(n_e) < g(n_e)$ and $n_e + 2 \leq n_{e+1}$ for all $e \in \mathbb{N}$, and two sequences (I_e) and (J_e) of rational numbers such that $I_e := [a_e; b_e]$ and $J_e := [c_e; d_e]$ satisfy the following conditions

$$a_e < b_e < c_e < d_e \ \& \ b_e - a_e = d_e - c_e = 2^{-(n_e+1)} \ \& \ c_e - b_e = 2^{-n_e}, \tag{4}$$

and $I_{e+1} \cup J_{e+1} \subset I_e$ for all $e \in \mathbb{N}$. The intervals I_e and J_e are reserved for the requirement R_e. That is, we construct a computable sequence (x_s) of rational numbers such that x_s is properly chosen from I_e or J_e in order to guarantee $\lim_s x_s \neq \lim_s \varphi_e(s)$. In general, the sequences (n_e), (I_e) and (J_e) is not computable but they can be effectively approximated. Namely, at stage s, we can construct the finite approximation sequence $(n_{e,s})_{e\leq k(s)}$, $(I_{e,s})_{e\leq k(s)}$ and $(J_{e,s})_{e\leq k(s)}$, where $k(s) \in \mathbb{N}$ satisfies $\lim_s k(s) = \infty$. At any stage s, we choose a rational number x_s such that $x_s \in I_{e,s}$ for all $e \leq k(s)$. If, for some t, $\varphi_{e,s}(t)$ enters the interval $I_{e,s}$ too, then we exchange $I_{e,s}$ and $J_{e,s}$. In this case, we denote this t by $t_{e,s}$. For any $i > e$, the intervals I_i and J_i will be cancelled and should be redefined with a new $n_{i,t} > n_{i,s}$ for some $t > s$. For the same n_e, the intervals I_e and J_e can be exchanged at most $f(n_e)$ times, if $(\varphi_e(s))_s$ converges f-effectively. We use $j_{e,s}$ to denote the number of such exchanges up to stage s for the current n_e.

The formal construction

Stage $s = 0$: Define $k(0) := 0$, $j_{0,0} := 0$, $n_{0,0} := (\mu n > 1)(f(n) < g(n)$ and $t_{0,0} := n_{0,0}$. Let $I_{0,0} := [a_{0,0}; b_{0,0}] := [0;\ 2^{-(n_{0,0}+1)}]$, $J_{0,0} := [c_{0,0}; d_{0,0}] := [3 \cdot 2^{-(n_{0,0}+1)};\ 2^{-n_{0,0}+1}]$ and $x_0 := 2^{-(n_{0,0}+2)}$. Notice that, x_0 is the middle point of the interval $I_{0,0}$.

Stage $s+1$: Given sequences $(n_{e,s})_{e\leq k(s)}$, $(t_{e,s})_{e\leq k(s)}$, $(j_{e,s})_{e\leq k(s)}$, $(I_{e,s})_{e\leq k(s)}$ and $(J_{e,s})_{e\leq k(s)}$. A requirement R_e *requires attention* if $e \leq k(s)$ and there is a $t > t_{e,s}$ such that

$$j_{e,s} \leq f(n_{e,s}) \ \& \ \varphi_{e,s}(t) \in I_{e,s}. \tag{5}$$

If no requirement requires attention, then define $x_{s+1} := x_s$ and $k(s+1) := k(s) + 1$. Furthermore, let

$$
\begin{cases}
n_{k(s+1),s+1} := (\mu n \geq n_{k(s),s} + 3)(f(n) < g(n)) \\
t_{k(s+1),s+1} := n_{k(s+1),s+1} \\
j_{k(s+1),s+1} := 0 \\
I_{k(s+1),s+1} := [x_s - 2^{-(n_{k(s+1),s+1}+2)}; x_s + 2^{-(n_{k(s+1),s+1}+2)}] \\
J_{k(s+1),s+1} := [x_s + 5 \cdot 2^{-(n_{k(s+1),s+1}+2)}; x_s + 7 \cdot 2^{-(n_{k(s+1),s+1}+2)}].
\end{cases} \tag{6}
$$

All other parameters remain the same as that of stage s. Notice that, in this case x_{s+1} is the middle point of the interval $I_{k(s+1),s+1}$ again.

Otherwise, let $e \leq k(s)$ be the minimal natural number such that R_e requires attention. Let x_{s+1} as the middle point of $J_{e,s}$ and $k(s+1) := e+1$. Define

$$
\begin{cases}
t_{e,s+1} := (\mu t > t_{e,s})(\varphi_{e,s}(t) \in I_{e,s}) \\
j_{e,s+1} := j_{e,s} + 1 \\
n_{e+1,s+1} := (\mu n \geq n_{k(s),s} + 3)(f(n) < g(n)) \\
j_{e+1,s+1} := 0 \\
t_{e+1,s+1} := n_{e+1,s+1} \\
I_{e,s+1} := J_{e,s} \\
J_{e,s+1} := I_{e,s} \\
I_{e+1,s+1} := [x_s - 2^{-(n_{e+1,s+1}+2)}; x_s + 2^{-(n_{e+1,s+1}+2)}] \\
J_{e+1,s+1} := [x_s + 5 \cdot 2^{-(n_{e+1,s+1}+2)}; x_s + 7 \cdot 2^{-(n_{e+1,s+1}+2)}].
\end{cases} \tag{7}
$$

All other parameters remain the same as that of stage s. We say that R_e receives attention at this stage. This completes the construction.

Now we verify our construction by the following sublemmas.

Sublemma 5.2.1 *For any $e, s \in \mathbb{N}$, we have $l(I_{e,s}) = l(J_{e,s}) = 2^{-(n_{e,s}+1)}$ and the distance between the intervals $I_{e,s}$ and $J_{e,s}$ is $2^{-n_{e,s}}$.*

Proof. This follows immediately from the construction. □ (sublemma)

Sublemma 5.2.2 *For any $e \in \mathbb{N}$, the requirement R_e receives attention at most finitely often. Thus, the limits $n_e := \lim_s n_{e,s}$, $I_e := \lim_s I_{e,s}$ and $J_e := \lim_s J_{e,s}$ exist. Furthermore, $\lim_e n_e = \lim_s k(s) = \infty$ holds.*

Proof. By induction on $e \in \mathbb{N}$. Suppose that R_i receives attention at most finitely often for any $i < e$. Let s_0 be the minimal stage s such that no requirement R_i ($i < e$) receives attention after stage s. Then there is a stage $s_1 \geq s_0$ at which we define n_{e,s_1} and the interval I_{e,s_1} and J_{e,s_1}. Since no requirement R_i for $i < e$ receives attention after stage s_1, we have $n_{e,t} = n_{e,s_1}$ for any $t \geq s_1$. Denote n_{e,s_1} simply by n_e. After stage s_1, R_e requires attention only if $\varphi_{e,s} \in I_{e,s}$ holds. If R_e receives attention, then the intervals I_e and J_e are exchanged. Since the distance between I_e and J_e is 2^{-n_e}, this can happen at most $f(n_e) + 1$ times. That is, R_e receives attention finitely often totally. Thus the rest of the sublemma follows directly. □ (sublemma)

Sublemma 5.2.3 *The limit $x := \lim_s x_s$ exists and x satisfies all the requirements R_e. Therefore x is not f-computable.*

Proof. By the construction, $x_s \in I_{e,s}$ for any $e \leq k(s+1)$ and $s \in \mathbb{N}$. Because, by Sublemma 5.2.2, the limit $I_e := \lim_s I_{e,s}$ exists, $x_s \in I_e$ if the intex s is big enough. On the other hand, it is easy to see that $\lim_e n_e = \infty$. Thus $\lim_e l(I_e) = 0$. Therefore, the limit $x := \lim_s x_s$ exists.

Now we are going to show that x satisfies all requirement R_e. Suppose that the sequence $(\varphi_e(s))_s$ f-effectively converges to some real number y_e. Let s_0 be the minimal stage s such that $k(s) = e$ and no requirements R_i for $i < e$ requires attention after stage s. Then $n_e := n_{e,s_0}$ is never changed after stage s. By the minimality of s_0, we have $j_{e,s_0} = 0$ and $t_{e,s_0} = -1$. If R_e requires no attention after stage s_0, then there is no $t \in \mathbb{N}$ such that $\varphi_e(t) \in I_{e,s_0}$. On the other hand, for any $s \geq s_0$ such that $n_{e+1,s} = n_{e+1}$, x_s has at lesst the distance $2^{-(n_{e+1}+1)}$ from the endpoints of $I_{e,s}$. Therefore $\lim_s \varphi_e(s) \neq x$.

Otherwise, suppose that R_e receives attention after stage s_0 at $s_1 < s_2 < s_3 < \cdots$. By a simple induction on $i \in \mathbb{N}$, we can show that $\varphi_e(t_{e,s_{i+1}}) \in I_{e,s_i}$ and $j_{e,s_i} = i$. Furthermore, the distance betbeen the intervals I_{e,s_i} and $I_{e,s_{i+1}}$ is 2^{-n_e}. Since $(\varphi_e(s))_s$ converges f-effectively, there must be an $i \leq f(n_e)$ such that $\varphi_e(t) \notin I_{e,s_{i+1}}$ for any $t \geq s_{i+1}$. Therefore $\lim_s \varphi_e(s) \neq x$ because x is an inner point of $I_e = I_{e,s_{i+1}}$. □ (sublemma)

Sublemma 5.2.4 *The sequence (x_s) converges g-effectively. Thus x is a g-computable real number.*

Proof. By the construction, for any $s < t$ and $n \in \mathbb{N}$, if $2^{-n} \leq |x_s - x_t| < 2^{-n+1}$, then $n = n_{e,u}$ for some $e, u \in \mathbb{N}$ and R_e receives attention at some stage between s and t with respect to the $n_{e,u}$. By a similar proof as that of Sublemma 5.2.3, we can show that R_e reseves attention at most $f(n_{e,u}) + 1 \leq g(n_{e,u})$ times with respect to the same $n_{e,u}$. This implies that the n-divergence of (x_s) is bounded by $g(n)$ for any $n \in \mathbb{N}$. Therefore $x := \lim_s x_s$ is a g-

computable real number. $\square$ (sublemma)

By Sublemma 5.2.3 and Sublemma 5.2.4, x is a g-computable but not f-computable real number. This completes the proof. $\square$

Corollary 5.3 *For any $k \in \mathbb{N}$, we have k-$\mathbf{EC} \subsetneq (k+1)$-$\mathbf{EC}$.*

Theorem 5.4 *There are $x, y \in$ 1-c$\mathbf{EC}$ such that $x - y \notin *$-c$\mathbf{EC}$. Therefore, k-c$\mathbf{EC}$ for $k > 0$ and $*$-c$\mathbf{EC}$ are not closed under the addition and subtraction.*

Proof. We will construct two computable increasing sequences (x_s) and (y_s) of rational numbers which converge 1-effectively to x and y, respectively, such that $z := x - y$ satisfies all the following requirements:

$$R_{\langle i,j\rangle}: \quad (\varphi_i(s))_s \text{ converges } j\text{-effectively to } u_i \Longrightarrow u_i \neq z$$

where (φ_i) is an effective enumeration of all partial computable functions $\varphi_i :\subseteq \mathbb{N} \to \mathbb{Q}$. To satisfy R_e ($e := \langle i, j\rangle$), we choose two natural numbers n_e and m_e such that $m_e = 2j+n_e+2$ and an rational interval $I := [a_e^0; a_e^8]$ of length 2^{-m_e+2}. The interval I is divided equally into eight subintervals $I_k := [a_e^k; a_e^{k+1}]$ for $k < 8$. At the beginning, let $x_0 := a_e^2$ and $y_0 = 0$ and hence $z_0 := x_0 - y_0 = a_e^2 \in J := I_e^2$, where J serves as a witness interval of R_e such that any element $z \in J$ satisfies R_e. If, at some stage $s_0 > 0$, $\varphi_i(t_0)$ enters the interval J for some t_0, then we define the $x_{s_0} := x_0 + 2^{-(n_e+1)} + 3 \cdot 2^{-(m_e+1)}$, $y_{s_0} := y_0 + 2^{-(n_e+1)}$ and $J := I_e^5$. Accordingly we have $z_{s_0} := x_{s_0} - y_{s_0} = z_0 + 3 \cdot 2^{-(m_e+1)}$ and hence $z_{s_0} \in J$. If, at a later stage $s_1 > s_0$, $\varphi_i(t_1)$ enters the interval $J := I_e^5$ for some $t_1 > t_0$, then we define the $x_{s_1} := x_{s_0} + 2^{-(n_e+2)} + 3 \cdot 2^{-(m_e+1)}$, $y_{s_1} := y_{s_0} + 2^{-(n_e+2)}$ and $J := I_e^2$. In this case, we have $z_{s_1} := x_{s_1} - y_{s_1} = z_0 + 3 \cdot 2^{-(m_e+1)}$ and hence $z_{s_1} \in J$. This can happen at most j times if $(\varphi_i(s))_s$ converges j-effectively. Thus we have $\lim_s z_s \neq \lim_s \varphi_i(s)$ and R_e is satisfied. To satisfy all the requirements, we apply a finite injury priority construction. $\square$

In the definition of h-Cauchy computability, we count the number of n-jumps which are between 2^{-n} and 2^{-n+1}. For the ω-Cauchy computability, we can simply count the number of jumps which are bigger than 2^{-n}. In other words, x is ω-Cauchy computable iff there are a computable function h and a computable sequence (x_s) of rational numbers converging to x such that, for any $n \in \mathbb{N}$, there are at most $h(n)$ non-nested pairs (i, j) of indices with $|x_i - x_j| \geq 2^{-n}$.

The next theorem gives another interesting characterization of ω-c$\mathbf{EC}$ where $\mathbb{CTF}$ is the class of all computable real functions $f : [0; 1] \to [0; 1]$.

Theorem 5.5 ω-c$\mathbf{EC} = \mathbb{CTF}(\mathbf{LC}) := \{f(y) : f \in \mathbb{CTF}\ \&\ y \in \mathbf{LC}\}$.

Proof. ($\supseteq$). Let (y_s) be an increasing computable sequence of rational numbers from $[0;1]$ which converges to y and $f \in \mathbb{CTF}$ with a computable modulus $e : \mathbb{N} \to \mathbb{N}$ such that

$$(\forall u, v \in [0;1])(\forall n \in \mathbb{N})(|u - v| \leq 2^{-e(n)} \Longrightarrow |f(u) - f(v)| \leq 2^{-n}). \tag{8}$$

We will show that $f(y) \in \omega$-c**EC**. By sequential computability of f, the sequence (x_s) defined by $x_s := f(y_s)$ is computable. Then, there is a computable sequence $(r_{s,t})$ of rational numbers such that $(\forall s, t \in \mathbb{N})(|x_s - r_{s,t}| \leq 2^{-(t+1)})$. Let $r_s := r_{s,(s+1)}$ for all $s \in \mathbb{N}$. Then $\lim_s r_s = \lim_s x_s = \lim_s f(y_s) = f(y)$. Since (y_s) is increasing, there are at most $2^{e(n+1)}$ pairs (i, j) of non-nested indices such that $|y_i - y_j| \geq 2^{-e(n+1)}$. By condition (8) and the inequality $|r_i - r_j| \leq |r_i - x_i| + |x_i - x_j| + |x_j - r_j| \leq |x_i - x_j| + 2^{-(n+1)}$ (for $i, j \geq n$), there are at most $2^{e(n+1)} + n$ pairs (i_s, j_s) of non-nested indices such that $|r_i - r_j| \geq 2^{-n}$. Thus, $f(y)$ is $(2^{e(n+1)} + n)$-cEC and hence $f(y) \in \omega$-c**EC**.

($\subseteq$). For computable function h_1 and computable sequence (x_s) of rational numbers which converges h_1-effectively to x, we construct a computable function $f : [0;1] \to [0;1]$ and a computable increasing sequence (y_s) of rational numbers converging to y such that $f(y) = x$. To this end, we define a computable sequence (p_s) of rational polygon functions which uniformly effectively converges to f.

Let $h(n) := \sum_{i=0}^{n} h_1(i)$. To define p_0, the interval $[0;1]$ is divided into subintervals of the same length by $0 = a_0 < a_1 < \cdots < a_{\delta(0)} = 1$ for $\delta(0) := 4h(1) + 1$. Then define p_0 as the polygon function which connects the points $(a_0, 0)$, $(a_{\delta(0)}, 1)$ and $(a_{4k+1}, 0)$, $(a_{4k+2}, 3/4)$, $(a_{4k+3}, 1/4)$, $(a_{4k+4}, 1)$ for every $k < h(1)$. Notice that, for the function p_0, we can easily define a computable increasing sequence (y_s^0) of rational numbers such that $|x_s - p_0(y_s^0)| \leq 3/4$ for all $s \in \mathbb{N}$. This can be done as follows. For $x_s \in [0;3/4]$, we choose a y_s^0 from some interval $[a_{4k+1}; a_{4k+2}]$. If for some $t_1 > s$, $x_{t_1} \notin [0;3/4]$ ($x_{t_1} \in [1/4;1]$ in this case), then choose a y_t^0 from the interval $[a_{4k+3}; a_{4k+4}]$. This implies obviously that $y_{t_1}^0 > y_s^0$. If $x_{t_2} \notin [1/4;1]$ for some $t_2 > t_1$, then choose a new $y_{t_2}^0$ from the interval $[a_{4(k+1)+1}; a_{4(k+1)+2}]$, and so on. Since $|x_{t_1} - x_{t_2}| \geq 2^{-1}$, this change can happen at most $h(1)$ times. So we can successfully choose y_s^0s increasingly.

The definition of p_1 equals to p_0 on each of the intervals $[a_{2k}; a_{2k+1}]$. On any interval $[a_{2k+1}; a_{2k+2}]$, we divide it first into $\delta(1) := 4h(2) + 1$ subintervals of the same length by $a_{2k+1} = b_0^k < b_1^k < \cdots < b_{\delta(1)}^k = a_{2k+2}$. If k is even, then, on the interval $[a_{2k}; a_{2k+1}]$, p_1 is the polygon function which connects all points $(b_0^k, 0)$, $(b_{\delta(1)}^k, 3/4)$ and $(b_{4j+1}^k, 0), (b_{4j+2}^k, 9/4^2), (b_{4j+3}^k, 3/4^2), (b_{4j+4}^k, 3/4)$ for all $j < h(2)$. Otherwise, if k is odd, p_1 connects all points $(b_0^k, 1/4)$, $(b_{\delta(1)}^k, 1)$, $(b_{4j+1}^k, 1/4)$, $(b_{4j+2}^k, 13/4^2)$, $(b_{4j+3}^k, 7/4^2)$ and $(b_{4j+4}^k, 1)$ for all $j < h(2)$. In general, the polygon function p_{n+1} can be defined inductively from p_n in a similar manner such that $|p_n(x) - p_{n+1}(x)| \leq (3/4)^n$ for any $x \in [0;1]$ and $n \in \mathbb{N}$.

Let $q_s := p_{3s}$ for any $s \in \mathbb{N}$. Then (q_s) is a computable sequence of rational polygon functions which converges uniformly effectively to a computable function f. For this function f, we can easily construct a computable increasing sequence (y_s) of rational numbers such that, for any s, $|f(y_s) - x_s| \leq 2^{-s}$. Thus, $x = \lim x_s = \lim f(y_s) = f(y) \in \mathbb{CTF}(\mathbf{LC})$. $\square$

References

[1] K. Ambos-Spies, K. Weihrauch, and X. Zheng. Weakly computable real numbers. *Journal of Complexity*, 16(4):676–690, 2000.

[2] A. J. Dunlop and M. B. Pour-El. The degree of unsolvability of a real number. In J. Blanck, V. Brattka, and P. Hertling, editors, *Computability and Complexity in Analysis*, volume 2064 of *LNCS*, pages 16–29, Berlin, 2001. Springer. CCA 2000, Swansea, UK, September 2000.

[3] Y. L. Ershov. A certain hierarchy of sets. i, ii, iii. (Russian). *Algebra i Logika.* 7(1):47–73, 1968; 7(4):15–47, 1968; 9:34–51, 1970.

[4] C.-K. Ho. Relatively recursive reals and real functions. *Theoretical Computer Science*, 210:99–120, 1999.

[5] K.-I. Ko. *Complexity Theory of Real Functions.* Progress in Theoretical Computer Science. Birkhäuser, Boston, 1991.

[6] J. Myhill. Criteria of constructibility for real numbers. *The Journal of Symbolic Logic*, 18(1):7–10, 1953.

[7] H. G. Rice. Recursive real numbers. *Proc. Amer. Math. Soc.*, 5:784–791, 1954.

[8] R. M. Robinson. Review of "Peter, R., Rekursive Funktionen". *The Journal of Symbolic Logic*, 16:280–282, 1951.

[9] J. R. Shoenfield. On degrees of unsolvability. *Ann. of Math. (2)*, 69:644–653, 1959.

[10] R. Soare. Cohesive sets and recursively enumerable Dedekind cuts. *Pacific J. Math.*, 31:215–231, 1969.

[11] R. I. Soare. *Recursively enumerable sets and degrees. A study of computable functions and computably generated sets.* Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1987.

[12] E. Specker. Nicht konstruktiv beweisbare Sätze der Analysis. *The Journal of Symbolic Logic*, 14(3):145–158, 1949.

[13] A. M. Turing. On computable numbers, with an application to the "Entscheidungsproblem". *Proceedings of the London Mathematical Society*, 42(2):230–265, 1936.

[14] X. Zheng. Recursive approximability of real numbers. *Mathematical Logic Quarterly*, 48(Suppl. 1):131–156, 2002.

[15] X. Zheng. On the Turing degrees of weakly computable real numbers. *Journal of Logic and Computation*, 13, 2003. (to appear).

[16] X. Zheng, D. Ding, and Z. Sun. On the definition of degrees of unsolvability for reals. *Chinese Science Bulletin*, 38(3):172–175, 1993. (in Chinese).