Computable Real Functions of Bounded Variation and Semi-Computable Real Numbers

(Extended Abstract)

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Abstract. In this paper we discuss some basic properties of computable real functions of bounded variation (CBV-functions for short). Especially, it is shown that the image set of semi-computable real numbers under CBV-functions is a proper subset of the class of weakly computable real numbers; Two applications of CBV-functions to semi-computable real numbers produce the whole closure of semi-computable real numbers under total computable real functions, and the image sets of semi-computable real numbers under monotone computable functions and CBV-functions are different.

1 Introduction

Continuity of a real function is one of the most important property in analysis. The effective counterpart of a continuous real function is the computable real function which can be computed by some algorithm with the respect to the effectively convergent Cauchy representation of real numbers. Computable real functions are widely discussed in literature, e.g., [4, 5, 11]. There are many problems, especially in applications to physical science, where more precise information about a function than continuity or computability are required. For example, it is very useful to be able to measure how rapidly a real function f oscillates on some interval $[a; b]$. However, the oscillatory character of a function is not easily determined from its continuity or even its computability. For this reason, the notion of the variation of a function was introduced in mathematics by Camille Jordan (1838–1922). Concretely, the variation $V_a^b(f)$ of f on the interval $[a; b]$ is defined as the supremum $\sup\left(\sum_{i<k} |f(x_i) - f(x_{i+1})|\right)$ which is taken over all possible subdivision $a = x_0 < x_1 < x_2 < \cdots < x_k = b$ of the interval (cf. [7]). This quantity turns out to be very useful for problems in physics, engineering, probability theory, Fourier series, and so forth.

A function f is called of *bounded variation* (BV for short) on an interval $[a; b]$, if the variation $V_a^b(f)$ of f on this interval is finite. Denote by $\mathbb{BV}[a; b]$ ($\mathbb{CBV}[a; b]$) the class of all (computable) real functions $f : [a; b] \to [a; b]$ which are of bounded variation on $[a; b]$. Especially, $\mathbb{BV}[0; 1]$ and $\mathbb{CBV}[0; 1]$ are denoted simply by $\mathbb{BV}$

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and $\mathbb{CBV}$, respectively. The class $\mathbb{BV}$ is widely discussed in classical mathematics. In this paper, we are more interested in the class $\mathbb{CBV}$. For example, we will discuss which classes of real numbers are closed under $\mathbb{CBV}$ and clarify the relationships among the image sets of $\mathbb{CBV}$ for different classes of real numbers.

Let's remind the definition of several interesting classes of real numbers discussed in effective analysis. A real number x is called *computable* if there is a computable sequence (x_s) of rational numbers which converges effectively to x in the sense that $|x - x_s| \leq 2^{-s}$ for any $s \in \mathbb{N}$; x is *left (right) computable* if there is an increasing (decreasing) computable sequence of rational numbers which converges to x; Left and right computable real numbers are called *semi-computable*; x is *weakly computable* if there are two left computable real numbers y and z such that $x = y - z$, and x is *recursively approximable* if there exists a computable sequence of rational numbers which converges to x. We denote by $\mathbf{EC}, \mathbf{LC}, \mathbf{RC}, \mathbf{SC}, \mathbf{WC}$ and $\mathbf{RA}$ the classes of computable, left computable, right computable, semi-computable, weakly computable and recursively approximable real numbers, respectively. These classes have been widely discussed in literature ($[1, 3, 8, 12, 14]$). Their relationship can be summarized as follows: $\mathbf{EC} = \mathbf{LC} \cap \mathbf{RC} \subsetneq \mathbf{SC} = \mathbf{LC} \cup \mathbf{RC} \subsetneq \mathbf{WC} \subsetneq \mathbf{RA}$. Besides, the classes $\mathbf{EC}, \mathbf{WC}$ and $\mathbf{RA}$ are algebraic fields, i.e., they are closed under the arithmetical operations $+, -, \times$ and $\div$. For $\mathbf{WC}$, another characterization is shown in [12] that, $x \in \mathbf{WC}$ iff there is a computable sequence (x_s) of rational numbers which converges weakly effectively to x in the sense that $\sum_{s\in\mathbb{N}} |x_s - x_{s+1}|$ is finite.

Obviously, the classes $\mathbf{EC}$ and $\mathbf{RA}$ are closed under the CBV-functions. Furthermore, the function f defined by $f(x) := 1 - x$ is a CBV-function which maps left computable real numbers to right computable ones and vice versa. Moreover, $g \circ f \in \mathbb{CBV}$ iff $g \in \mathbb{CBV}$ for any g. This observation implies that $\mathbb{CBV}(\mathbf{LC}) = \mathbb{CBV}(\mathbf{RC}) = \mathbb{CBV}(\mathbf{SC})$, where $\mathbb{CBV}(\mathbf{C}) := \{g(x) : x \in \mathbf{C} \; \& \; g \in \mathbb{CBV}\}$ denotes the image set of $\mathbf{C}$ under functions of $\mathbb{CBV}$. In [6], it is shown that both the classes $\mathbf{SC}$ and $\mathbf{WC}$ are not closed under $\mathbb{CBV}$ and that the image of a semi-computable real number under a CBV-function is weakly computable. In other words, $\mathbf{SC} \subsetneq \mathbb{CBV}(\mathbf{SC}) \subseteq \mathbf{WC}$ and $\mathbf{WC} \subsetneq \mathbb{CBV}(\mathbf{WC})$ hold. On the other hand, it is also shown in [6] that $\mathbf{WC} \subsetneq \mathbb{CTF}(\mathbf{SC}) = \mathbb{CTF}(\mathbf{WC}) \subsetneq \mathbf{RA}$, where $\mathbb{CTF}$ is the set of all computable total real functions $f : [0; 1] \to [0; 1]$. Namely, the image sets of semi-computable and weakly computable real numbers under total computable real functions are the same and they locate strictly between the classes $\mathbf{WC}$ and $\mathbf{RA}$. Two interesting questions remain open. That is, whether $\mathbb{CBV}(\mathbf{SC}) = \mathbf{WC}$? and $\mathbb{CBV}(\mathbf{WC}) = \mathbb{CTF}(\mathbf{WC})$? For the first question, we will show a negative answer. A positive answer to the second question follows from the stronger result $\mathbb{CBV}^2(\mathbf{SC}) = \mathbb{CTF}(\mathbf{SC})$. This result shows that any application of a total computable real function to a semi-computable real number can be realized by two consecutive applications of the CBV-functions to some (possibly different) semi-computable real number. Finally, we will show that the image sets of weakly computable real numbers under computable monotone functions and under usual total computable real functions are different.

2 Preliminaries

In this section we will recall some known results, notions and notations which will be used later.

Let Σ be any alphabet. Σ^* and Σ^ω are the sets of all finite strings and infinite sequences of Σ, respectively. For $u, v \in \Sigma^*$, denote by uv the concatenation of v after u. u is an initial segment of w (denoted by $u \sqsubseteq w$) if $w = uv$ for some v and $u \sqsubset w$ means $u \sqsubseteq w$ & $w \neq w$. If $w \in \Sigma^* \cup \Sigma^\omega$, then $w[n]$ denotes its n-th element. Thus, $w = w[0]w[1] \cdots w[n-1]$, if $|w|$, the length of w, is n, and $w = w[0]w[1]w[2] \cdots$, if $|w| = \infty$. The unique string of length 0 is denoted by λ (so-called empty string). For any finite string $w \in \{0;1\}^*$, and number $n \leq |w|$, the restriction $w \upharpoonright n$ is defined by $(w \upharpoonright n)[i] := w[i]$ if $i < n$ and $(w \upharpoonright n)[i] := \uparrow$, otherwise. Then the length $|w \upharpoonright n| = n$.

We denote by $\mathbb{N}, \mathbb{Q}$ and $\mathbb{R}$ the sets of all natural, rational and real numbers, respectively. $[0;1]_{\mathbb{Q}}$ is the set of all rational numbers $x \in [0;1]$. For any sets A and B, $f :\subseteq A \to B$ is a partial function such that $\mathrm{dom}(f) \subseteq A$ and $\mathrm{range}(f) \subseteq B$, while $f : A \to B$ denotes a total function from A to B, i.e., $\mathrm{dom}(f) = A$. If $I \subset \mathbb{R}$ is an interval, then its length is denoted by $l(I)$.

The computability notions on subsets $A \subseteq \mathbb{N}$ and functions $f :\subseteq \mathbb{N}^k \to \mathbb{N}$ are well defined and developed in classical computability theory (cf. [9, 10]). For other countable set, say, $\mathbb{Q}$, the corresponding notions of computability can be defined accordingly by means of some coding. For example, if $\langle \cdot, \cdot \rangle : \mathbb{N}^2 \to \mathbb{N}$ is a computable pairing function, then we can define a coding $\sigma : \mathbb{N} \to \mathbb{Q}$ by $\sigma(\langle \langle n, m \rangle, k \rangle) := (n - m)/(k + 1))$ for any $n, m, k \in \mathbb{N}$ and call a set $A \subseteq \mathbb{Q}$ recursive or recursively enumerable if the set $\sigma^{-1}(A) := \{n \in \mathbb{N} : \sigma(n) \in A\}$ is recursive or recursively enumerable, respectively. A function $f :\subseteq \mathbb{N} \to \mathbb{Q}$ is computable if there is a computable function $g :\subseteq \mathbb{N} \to \mathbb{N}$ such that $f(n) = \sigma \circ g(n)$ for any $n \in \mathrm{dom}(\sigma \circ g)$, and so forth. Especially, we call a sequence (x_s) of rational numbers computable if there is a computable function $f : \mathbb{N} \to \mathbb{Q}$ such that $x_s = f(s)$ for any s.

The computability of real functions can be defined by type-2 Turing machines (Weihrauch [10, 11]). A type-2 Turing machine M extends the classical Turing machine in such a way that it accepts infinite sequences as well as finite strings as inputs and outputs. For any input p, $M(p)$ outputs a finite string q if $M(p)$ halts after finite steps with q in its write-only output tape, while $M(p)$ outputs an infinite sequence q means that $M(p)$ will never halt and keep writing the sequence q on its output tape. A real function $f :\subseteq \mathbb{R} \to \mathbb{R}$ is *computable*, if there is a type-2 Turing machine M which computes f in the sense that, for any $x \in \mathrm{dom}(f)$ and any sequence p (of rational numbers) which converges effectively to x, $M(p)$ outputs a sequence q (of rational numbers) which converges effectively to $f(x)$. Therefore, any computable function is continuous on its domain.

The closure properties of the real number classes mentioned in the last section under the class $\mathbb{CPF}$ of computable partial real functions are first discussed in [13]. These discussions are extended to the case of $\mathbb{CTF}$ in [6], where the divergence bounded computability is also introduced. For any sequence (x_s) and $n \in \mathbb{N}$. The n-divergence of (x_s) is defined as the maximal m such that, for some chain $i_0 < j_0 \leq i_1 < j_1 \leq \cdots \leq i_m < j_m$ of natural numbers, $|x_{i_s} - x_{j_s}| \geq 2^{-n}$ holds for any $s \leq m$. A real number x is called *divergence bounded computable* if there is a recursive function h and a computable sequence (x_s) of rational numbers converging to x such that the n-divergence of (x_s) is bounded by $h(n)$ for any n. The class of all divergence bounded computable real numbers is denoted by **DBC**. We can summarize some of the main results of [13, 6] as follows.

Theorem 2.1 (Zheng [13] and Rettinger et al. [6]).

1. $\mathbb{CPF}(\mathbf{LC}) = \mathbb{CPF}(\mathbf{RC}) = \mathbb{CPF}(\mathbf{RA}) = \mathbf{RA}$.
2. $\mathbb{CTF}(\mathbf{LC}) = \mathbb{CTF}(\mathbf{RC}) = \mathbb{CTF}(\mathbf{WC}) = \mathbf{DBC}$.
3. $\mathbf{WC} \subsetneqq \mathbb{CBV}(\mathbf{WC}) \subseteq \mathbb{CTF}(\mathbf{WC}) \subsetneqq \mathbf{RA}$.
4. $\mathbf{SC} \subsetneqq \mathbb{CBV}(\mathbf{LC}) = \mathbb{CBV}(\mathbf{RC}) = \mathbb{CBV}(\mathbf{SC}) \subseteq \mathbf{WC}$.

Let $\delta : \mathbb{N} \to \mathbb{N}^+$ and $\mathbb{N}^*_\delta := \{w \in \mathbb{N}^* : \forall n < |w|\, (w[n] < \delta(n))\}$. We define a *$\delta$-interval tree* ($\delta$-i.t., for short) on $[0;1]$ as a function $I : \mathbb{N}^*_\delta \to \mathbb{I}$, where $\mathbb{I}$ is the set of all rational intervals on $[0;1]$, such that $I(\lambda) = [0;1]$; $\bigcup_{i<\delta(|w|)} I(wi) = I(w)$, for any $w \in \mathbb{N}^*_\delta$ and $\lim_{s\to\infty} l(I(w_s)) = 0$ for any sequence (w_s) of $\mathbb{N}^*_\delta$ with $w_s \sqsubset w_{s+1}$. For any δ-i.t. I and $w \in \mathbb{N}^*_\delta$, the interval $I(w)$ is denoted by $I(w) := [a^\delta_w; b^\delta_w]$. A δ-i.t. I is called *computable* if the functions $a, b : \mathbb{N}^*_\delta \to [0;1]_{\mathbb{Q}}$ defined by $a(w) := a^\delta_w$ and $b(w) := b^\delta_w$, respectively, are computable. A δ-i.t. I is called *canonical* if, for any $w \in \mathbb{N}^*_\delta$, the interval $I(w)$ is divided into subintervals $I(w0), I(w1), \cdots, I(w(\delta(|w|)-1))$ disjunctively and equally, in other words, $a^\delta_w := \sum_{i<|w|} \left(w[i] \cdot \prod_{j\leq i} \delta(j)^{-1} \right)$ and $b^\delta_w := a^\delta_w + \prod_{j<|w|} \delta(j)^{-1}$.

Furthermore, for any $\delta_1, \delta_2 : \mathbb{N} \to \mathbb{N}^+$, a function $\iota :\subseteq \mathbb{N}^*_{\delta_1} \to \mathbb{N}^*_{\delta_2}$ is called *(δ_1, δ_2)-compatible* if the domain $\mathrm{dom}(\iota)$ of ι is infinite and alternate in the sense that, $w(i-1), w(i+1) \notin \mathrm{dom}(\iota)$ & $i \neq 0, \delta_1$ for any $w \in \mathbb{N}^*_{\delta_1}$ and $i < \delta_1(|w|)$ such that $wi \in \mathrm{dom}(\iota)$; $\forall w, v \in \mathbb{N}^*_{\delta_1}\, (w \in \mathrm{dom}(\iota)\ \&\ v \sqsubseteq w \Longrightarrow v \in \mathrm{dom}(\iota)\ \&\ \iota(v) \sqsubseteq \iota(w))$) and $\forall u, v \in \mathrm{dom}(\iota)\, (|u| = |v| \Longrightarrow |\iota(u)| = |\iota(v)|)$.

The most important application of this notion is the following technical lemma which is very useful to construct some computable real functions.

Lemma 2.2 (Rettinger et al. [6]). *Let $\delta_1, \delta_2, e : \mathbb{N} \to \mathbb{N}^+$ be computable functions, I_1 a canonical δ_1-i.t., and I_2 a computable δ_2-i.t. with $l(I(w)) \leq 2^{-e(|w|)}$ for all $w \in \mathbb{N}^*_{\delta_2}$. If $\iota :\subseteq \mathbb{N}^*_{\delta_1} \to \mathbb{N}^*_{\delta_2}$ is a (δ_1, δ_2)-compatible computable function, then there is a computable function $f : [0;1] \to [0;1]$ such that $f(I_1(w)) \subseteq I_2(\iota(w))$ and $f(a^{\delta_1}_w) = a^{\delta_2}_{\iota(w)}$ for all $w \in \mathrm{dom}(\iota)$.*

3 Computable Functions of Bounded Variation

In this section, we will discuss some basic properties of $\mathbb{CBV}$. Especially, we investigate which properties of continuous functions of bounded variation can be extended to that of $\mathbb{CBV}$ accordingly. For the BV-functions, we have at first the following simple properties which hold obviously for the CBV-functions too.

Proposition 3.1. *1. If $f, g \in \mathbb{BV}$, then $f+g, f-g, f \cdot g \in \mathbb{BV}$. If, in addition, $(\exists c > 0)(\forall x \in [0;1])(|g(x)| \geq c)$ holds, then $f/g \in \mathbb{BV}$;*

2. Let $L_a^b(f)$ denote the length of the graph of function f on the interval $[a;b]$, then $V_a^b(f) + (b-a) \geq L_a^b(f) \geq [(V_a^b(f))^2 + (b-a)^2]^{1/2}$. Therefore, $f \in \mathbb{BV}[a;b]$ iff the graph of f on $[a;b]$ has finite length.

3. There are $f, g \in \mathbb{BV}$ such that $f \circ g \notin \mathbb{BV}$, i.e., $\mathbb{BV}$ is not closed under composition.

Some other properties of $\mathbb{CBV}$ are summarized in the following lemma.

Lemma 3.2. *1. If $f \in \mathbb{CBV}$, then $V_0^1(f) = \sup_{\mathbb{Q}} \left(\sum_{i<m} |f(r_i) - f(r_{i+1})| \right)$, where the supremum $\sup_{\mathbb{Q}}$ is taken over all rational subdivision $0 = r_0 < r_1 < r_2 < \cdots < r_m = 1$ for $r_i \in \mathbb{Q}$.*

2. If both $f : [0;1] \to [0;1]$ and its first order derivative f' are computable, then $f \in \mathbb{CBV}$ and v_f is a computable function, where $v_f(x) := V_0^x(f)$.

3. For $f \in \mathbb{CBV}$, the variation $V_0^1(f)$ is a left computable real number. And for any $y \in \mathbf{LC}$, there is a function $f \in \mathbb{CBV}$ such that $V_0^1(f) = y$.

Classically, for any $f \in \mathbb{BV}$, there are nondecreasing functions g, h such that $f(x) = g(x) - h(x)$ for all $x \in [0;1]$. Moreover, if f is continuous, then g, h can also be chosen to be continuous. Unfortunately, the result cannot be extended immediately to $\mathbb{CBV}$as shown in [15]. However this claim can still be true if we require that $V_0^1(f)$ is computable as well. This observation belongs essentially to Douglas Bridges [2].

Theorem 3.3 (Bridges [2]). *Let $f \in \mathbb{CBV}$. If $V_0^1(f)$ is computable, then there are two computable nondecreasing function $g, h : [0;1] \to [0;1]$ such that $f(x) = g(x) - h(x)$ for any $x \in [0;1]$.*

4 $\mathbb{CBV}$(SC) and WC

By Theorem 2.1.4, the image of a semi-computable real number under a CBV-function is weakly computable. In this section we will show that not every weakly computable real number is such an image. To this end, let's look at an important property of CBV-functions.

Given an interval $J \subseteq [0;1]$ of length δ and a continuous function $f : [0;1] \to [0;1]$, a pair (x_1, x_2) of real numbers is called a crossing of f over J if $f(x_1)$ and $f(x_2)$ locate on different sides of the interval J. Denoted by $z(f, J)$ the number of crossings of f over J, namely

$$\begin{aligned} z(f,J) := \max\{n \in \mathbb{N} : (\exists (x_i)_{i \leq n})(0 \leq x_0 < x_1 < \cdots < x_n \leq 1 \ \& \\ (\forall i < n)((x_i, x_{i+1}) \text{ is a crossing of } f \text{ over } J))\}. \end{aligned}$$

If $f \in \mathbb{CBV}$ and $[0;1]$ is divided equally into n-subintervals J_i of length $1/n$ for $i < n$. Then $\sum_{i<n} z(f, J_i)/n \leq V_a^b(f)$ holds for any $n \in \mathbb{N}$. This implies that, $(\forall e \in \mathbb{N})(\exists n \in \mathbb{N})(\exists i < n)(z(f, J_i)/n \leq 2^{-e})$. Because this observation is very essential for the proof of the following Theorem 4.2, we state it as a separate lemma.

Lemma 4.1. *Let $f \in \mathbb{CBV}$ and $I \subseteq [0;1]$. For any $e \in \mathbb{N}$, there are $\delta > 0$ and $a_0, a_1, a_2, a_3 \in I$ such that the intervals $I_i := (a_i; a_{i+1})$ have the same length δ for $i < 3$ and $z(f, I_1) \cdot 3 \cdot \delta \leq 2^{-e}$.*

Theorem 4.2. *$\mathbb{CBV}(\mathbf{LC}) \subsetneq \mathbf{WC}$.*

Proof. (sketch) The inclusion part is quite straightforward. Here we prove only the inequality part. Namely we show that there is a weakly computable real number y such that $y \neq f(x)$ for any $x \in \mathbf{LC}$ and any $f \in \mathbb{CBV}$.

Let (φ_s) and (γ_s) be effective enumerations of computable functions $\varphi_s :\subseteq \mathbb{R} \to \mathbb{R}$, and $\gamma_s :\subseteq \mathbb{N} \to \mathbb{Q}$, respectively. It suffices to construct effectively a computable sequence (y_s) of rational numbers which converges weakly effectively to some y and y satisfies, for any $i, j \in \mathbb{N}$, the requirement

$$R_{\langle i,j\rangle}: \quad \varphi_i \in \mathbb{CBV} \ \& \ \forall s(\gamma_j(s) \leq \gamma_j(s+1)) \Longrightarrow y \neq \lim_{s\to\infty} \varphi_i(\gamma_j(s))).$$

The strategy to satisfy a single requirement R_e ($e = \langle i,j\rangle$) is simple. For example, we can fix a rational interval $I \subseteq [0;1]$ as a base interval and choose arbitrarily two open subintervals I_1 and I_2 of I such that they have at least a positive distance. Then, one of these subintervals is a witness interval of R_e in the sense that each element of this interval satisfies R_e. Actually, this witness interval can be determined in finite steps, if $\varphi_i \in \mathbb{CBV}$ & $\forall s(\gamma_j(s) \leq \gamma_j(s+1))$ holds. Namely, we choose at first I_1, then change to I_2 if some $\varphi_i(\gamma_j(s))$ enters I_1, and change to I_1 again whenever $\varphi_i(\gamma_j(s))$ enters I_2 for a larger s, and so on. If the limit $\lim_{s\to\infty} \varphi_i(\gamma_j(s))$ exists, we can change the interval only finitely often and the last interval we have chosen is the witness interval of R_e. Otherwise, if the limit does not exist, then both I_1 and I_2 are witness intervals of R_e.

To satisfy all requirements R_e simultaneously, we will try to find a sequence (I_s) of nested open intervals such that $I_{s+1} \subsetneq I_s$. For each s, I_s and I_{s+1} are base and witness interval of R_s, respectively. If we require in addition that $\lim_{s\to\infty} l(I_s) = 0$, and define $y_s := \mathrm{mid}(I_s)$ (the middle point of I_s), then the sequence (y_s) converges to a limit y which belongs to all intervals I_e and hence satisfies all requirements R_e. To ensure that the sequence (y_s) converges weakly effectively, we choose the witness intervals in such a way that $l(I_s) \leq 2^{-s}$ for any $s \in \mathbb{N}$. This implies that $|y_s - y_{s+1}| \leq 2^{-s}$ and hence $\sum_{s\in\mathbb{N}} |y_s - y_{s+1}| \leq 2$.

Unfortunately, the sequence (y_s) mentioned above is not computable, because the sequence (I_s) of witness intervals is not computable. However we can construct one of its effective approximation $(I_{e,s})_{e<d_s, s\in\mathbb{N}}$ such that $\lim_{s\to\infty} d_s = \infty$, $I_{e,s}$ and $I_{e+1,s}$ are current base and witness intervals at stage s, respectively, of the requirement $R_{\langle i,j\rangle}$ for $(\varphi_{i,s}, \gamma_{j,s})$ instead of (φ_i, γ_j). At the same time, define $y_s := \mathrm{mid}(I_{d_s,s})$. Of course, the "injury" phenomenon could appear in this construction. For example, given $(I_{e,s})_{e\leq d_s}$, we might define a new witness interval $I_{e_1,s+1}$ for some $e_1 < d_s$ at stage $s+1$. In this case, all $I_{e,s}$ for $e_1 < e \leq d_s$ are destroyed and have to be redefined later again. We say that the corresponding requirements R_e is injured (by R_{e_1}). Fortunately, any requirement R_e can be injured finitely often and its witness interval $I_e := \lim_{s\to\infty} I_{e,s}$ exists.

Nevertheless, the injury in the above construction introduces also extra jumps of the sequence (y_s). To guarantee that the sum $\sum_{s\in\mathbb{N}} |y_s - y_{s+1}|$ is still finite, more efforts are needed. Concretely, given a current base interval $I_{e,s}$ of R_e, we choose at stage $s+1$, according to Lemma 4.1, three subintervals I_0, I_1 and I_2 of length δ such that $z(f, I_1)\cdot 3\delta \leq 2^{-(e+b_{e,s})}$ where $b_{e,s}$ is the number of injuries that R_e received up to stage s and let I_0 or I_2 be the witness interval of R_e. This implies that, R_e contributes to the jumps of (y_s) hereafter at most $2^{-(e+b_{e,s})}$ whenever it is not injured again. Generally, let S_e be the set of all e-stages s at which y_s is defined according to R_e. Then, we have $\sum_{s+1\in S_e} |y_s - y_{s+1}| \leq 2^{-e}$. In other words, the e-stages contribute at most 2^{-e} to the sum $\sum_{s+1\in\mathbb{N}} |y_s - y_{s+1}|$. Therefore, $\sum_{s\in\mathbb{N}} |y_s - y_{s+1}| = \sum_{e\in\mathbb{N}} \sum_{s+1\in S_e} |y_s - y_{s+1}| \leq \sum_{e\in\mathbb{N}} 2^{-e} \leq 2$. That is $y := \lim_{s\to\infty} y_s$ is a weakly computable real number which satisfies all requirements R_e. Thus, $x \in \mathbf{WC}/\mathbb{CBV}(\mathbf{LC})$.

Corollary 4.3. $\mathbb{CBV}(\mathbf{LC}) \subsetneq \mathbb{CTF}(\mathbf{LC})$.

5 $\mathbb{CBV}^2(\mathbf{LC})$ and $\mathbb{CTF}(\mathbf{LC})$

In the last section we have shown that one application of CBV-functions to **SC** produces a proper subset of **WC**. However, we will show in this section that two applications of CBV-functions suffice to produce the set $\mathbb{CTF}(\mathbf{LC})$. In the following, let $\mathbb{CBV}^2 := \{f\circ g : f, g \in \mathbb{CBV}\}$.

Theorem 5.1. $\mathbb{CBV}^2(\mathbf{LC}) = \mathbb{CTF}(\mathbf{LC}) = \mathbf{DBC}$

Proof. (sketch) The inclusion $\mathbb{CBV}^2(\mathbf{LC}) \subseteq \mathbb{CTF}(\mathbf{LC})$ is trivial because $\mathbb{CBV}^2 \subseteq \mathbb{CTF}$. We prove now that $\mathbf{DBC} \subseteq \mathbb{CBV}^2(\mathbf{LC})$.

Given a $y \in \mathbf{DBC}$, there is a recursive function $b : \mathbb{N} \to \mathbb{N}$ and a computable sequence (y_s) of rational numbers converging to y such that, for any $n \in \mathbb{N}$, the n-divergence of (y_s) is bounded by $b(n)$. Assume w.l.o.g. that $b(n) \geq 1$. We will construct two computable functions $g, h \in \mathbb{CBV}$ and an increasing computable sequence (x_s) of rational numbers converging to x such that $g \circ h(x) = y$.

By definition, for any $f \in \mathbb{CBV}$, f can only have few big jumps or many small jumps. Since $\mathbb{CBV}(\mathbf{LC}) \neq \mathbb{CTF}(\mathbf{LC})$, the composition $g\circ h$ cannot be of bounded variation. That is, to satisfy $gh(x) = y$, the function $g\circ h$ should have a lot of big jumps but g and h should not. The essential idea is that, we let h have a lot of small jumps and then let g amplify them to the big ones.

Let $\delta_1(n) := 2b(3n) + 1$, $\delta_2(n) := 2^{n+1} \cdot \prod_{i\leq n}(b(3i) + 1) + 1$ and $\delta_3(n) := 2$ be computable functions. I_1 and I_2 are canonical δ_1- and δ_2-interval trees, respectively. We define I_3 as a δ_3-interval tree in such a way that, for any $w \in \{0,1\}^*$ and $i \in \{0, 1\}$, the interval $I_3(w)$ is covered by intervals $I_3(w0)$ and $I_3(w1)$ which are overlapped in the middle of the interval $I_3(w)$ for a length of $2^{-3|w|+1}$. More precisely, the interval $I_3(w) := [a^{\delta_3}_w; b^{\delta_3}_w]$ has the length of $l^{\delta_3}_w := \prod_{i<|w|}(2^{-1} + 2^{-3i})$ for $a^{\delta_3}_w, b^{\delta_3}_w$ defined by

$$a^{\delta_3}_w := \sum_{i<|w|} w[i]\cdot(2^{-1} + 2^{-3(i+1)})\cdot\prod_{j<i}(2^{-1} + 2^{-3j}) \quad \text{and}$$

$$b_w^{\delta_3} := \sum_{i<|w|} w[i] \cdot (2^{-1} + 2^{-3(i+1)}) \cdot \prod_{j<i}(2^{-1} + 2^{-3j}) + \prod_{i<|w|} (2^{-1} + 2^{-3i}).$$

Furthermore, we define two functions $\iota_1 : \mathbb{N}^*_{\delta_1} \to \mathbb{N}^*_{\delta_2}$ and $\iota_2 : \mathbb{N}^*_{\delta_2} \to \mathbb{N}^*_{\delta_3}$ inductively by $\iota_1(\lambda) := \lambda$, $\iota_2(\lambda) := \lambda$ and

$$\begin{aligned} \iota_1(wi) &:= (\iota_1(w)1 \text{ if } \exists j(i = 4j+1); \iota_1(w)3 \text{ if } \exists j(i = 4j+3); \uparrow \text{ otherwise}) \\ \iota_2(wi) &:= (\iota_2(w)0, \text{ if } i = 1; \iota_2(w)1 \text{ if } i = 3; \uparrow \text{ otherwise }) \end{aligned}$$

for any $w \in \mathbb{N}^*$. Obviously, both ι_1 and ι_2 are computable functions. They are also (δ_1, δ_2)- and (δ_2, δ_3)-compatible, respectively. By Lemma 2.2, there are computable real functions $g, h : [0; 1] \to [0; 1]$ such that

$$h(I_1(w)) \subseteq I_2(\iota_1(w)) \ \& \ g(I_2(u)) \subseteq I_3(\iota_2(u)) \tag{1}$$

for any $w \in \mathrm{dom}(\iota_1)$ and $u \in \mathrm{dom}(\iota_2)$. It is not difficult to see that both g and h are of finite variations. It remains to construct an increasing computable sequence (x_s) of rational numbers. Notice that, for any $w \in \mathrm{dom}(\iota_1)$, $\iota_1(w) \in \mathrm{dom}(\iota_2)$ and hence $gh(a_w^{\delta_1}) = g(a^{\delta_2}_{\iota_1(w)}) = a^{\delta_3}_{\iota_1\iota_2(w)}$. We define $x_s := a^{\delta_1}_{w_s}$ where (w_s) is a computable sequence in $\mathbb{N}^*_{\delta_1}$ which is defined as follows.

Stage $s = 0$. Define simply $w_0 = \lambda$ and hence $x_0 := 0$.

Stage $s+1$. Given $w_s \in \mathrm{dom}(\iota_1)$. If $y_s \in I_3(\iota_2\iota_1(w_s))$, then define $w_{s+1} := w_s1$ if $y_s \in I_3(\iota_2\iota_1(w_s1))$ and $w_{s+1} := w_s3$ otherwise.

Suppose now that $y_s \notin I_3(\iota_1\iota_2(w_s))$. Then choose an $n \leq |w_s|$ such that $y_s \in I_3(\iota_1\iota_2(w_s) \upharpoonright n)$, $y_s \notin I_3(\iota_1\iota_2(w_s) \upharpoonright (n+1))$ and let $w_{s+1} = (w_s \upharpoonright n)(w_s[n] + 2)$. Notice that, $\iota_1(w_{s+1})[n] = 1$ if $\iota_1(w_s)[n] = 3$ and $\iota_1(w_{s+1})[n] = 3$ otherwise. This implies that $\iota_2\iota_1(w_{s+1})[n] = 1 \dot{-} \iota_2\iota_1(w_s)[n]$, and hence $y_s \in I_3(\iota_2\iota_1(w_{s+1})[n])$.

We can show that (x_s) constructed above is an increasing computable sequence and $\lim_{s\to\infty} gh(x_s) = \lim_{s\to\infty} y_s = y$.

Corollary 5.2. $\mathbb{CBV}(\mathbf{WC}) = \mathbb{CTF}(\mathbf{WC})$

Proof. This follows immediately from Theorem 2.1 and Theorem 5.1.

6 $\mathbb{CMF}(\mathbf{WC})$ and $\mathbb{CTF}(\mathbf{WC})$

In this section we will show that the image sets of **WC** under $\mathbb{CTF}$ and $\mathbb{CMF}$ are different, where $\mathbb{CMF}$ is the class of computable monotone functions $f : [0; 1] \to [0; 1]$. Notice first that, $y \in \mathbb{CMF}(\mathbf{WC})$ iff there is an $x \in \mathbf{WC}$ and a strictly monotone and computable function f such that $y = f(x)$. For strictly monotone functions, we have the following useful lemma.

Lemma 6.1. *Let $f : [0; 1] \to [0; 1]$ be a strictly monotone function, $J \subseteq [0; 1]$ a non-empty rational interval. There is a $t_0 \in \mathbb{N}$ such that, for any $r \geq t_0$, there are rational numbers $a_1 < a_2 < a_3 < a_4$ which belong to the interval J and satisfy the following condition:*

$$a_3 - a_2 \geq 2^{-(2r+2)} \ \& \ |f(a_1) - f(a_4)| \leq 2^{-(r+1)}. \tag{2}$$

Theorem 6.2. $\mathbb{CMF}(\mathbf{WC}) \subsetneq \mathbb{CBV}(\mathbf{WC}) = \mathbf{DBC}$

Proof. It suffices to prove the inequality part. We will construct a recursive function h and a computable sequence (y_s) of rational numbers converging to y so that the n-divergence of (y_s) is bounded by $h(n)$ for any n. Thus $y \in \mathbf{DBC} = \mathbb{CTF}(\mathbf{WC})$. Furthermore, y satisfies, for all $i, j \in \mathbb{N}$, the requirement

$$R_{\langle i,j\rangle}: \quad \begin{array}{l} \text{If } \varphi_i \text{ is a strictly monotone total function and } \gamma_j \text{ is total such} \\ \text{that } \sum_{s\in\mathbb{N}} |\gamma_j(s)-\gamma_j(s+1)| \leq 1, \text{ then } \lim_{s\to\infty} \varphi_i(\gamma_j(s)) \neq y, \end{array}$$

where (φ_e) and (γ_e) are effective enumerations of all computable functions $\varphi_e :\subseteq [0;1] \to [0;1]$ and $\gamma_e :\subseteq \mathbb{N} \to [0;1]_{\mathbb{Q}}$. Thus, $y \notin \mathbb{CMF}(\mathbf{WC})$.

Given a strictly monotone function φ_i and a weakly convergent sequence $(\gamma_j(s))$ with $\sum_{s\in\mathbb{N}} |\gamma_j(s) - \gamma_j(s+1)| \leq 1$, we consider a base interval $J \subseteq [0;1]$. Let r and a_1, a_2, a_3, a_4 satisfy Lemma 6.1 and define $I^1 := [a_1; a_2]$, $I^2 := [a_3; a_4]$, $J^1 := \varphi_i(I^1)$ and $J^2 := \varphi_i(I^2)$. Now, if $\varphi_i(\gamma_j(s))$ enters J^1 (hence $\gamma_j(s)$ enters I^1), then we define y_{s+1} as the middle point of J^2. Similarly, if $\varphi_i(\gamma_j(s))$ enters J^2 (hence $\gamma_j(s)$ enters I^2), then we define y_{s+1} to be the middle point of J^1. This guarantees that the limits $y := \lim_{s\to\infty} y_s$ and $\lim_{s\to\infty} \varphi_i(\gamma_j(s))$ have at least a distance of $|\varphi_i(a_2) - \varphi_i(a_3)|$, hence y satisfies the requirement $R_{\langle i,j\rangle}$. Notice that the y_s's can be redefined according to this strategy at most 2^{2r+2} times because $a_3 - a_2 \geq 2^{-(2r+2)}$ and $\sum_{s\in\mathbb{N}} |\gamma_j(s) - \gamma_j(s+1)| \leq 1$. On the other hand, every redefinition of y_s contributes only a jump which is bounded by $2^{-(r+1)}$ because of (2).

To satisfy all requirements R_e simultaneously, let's begin with the base interval $I_0 := [0;1]$ and search for the minimal $e := \langle i, j\rangle$ such that we can apply Lemma 6.1 for the function φ_i. Choose r_1 and a_1, a_2, a_3, a_4 which satisfy Lemma 6.1 and let $I_e^1 := [a_1; a_2]$, $I_e^2 := [a_3; a_4]$ and $J_e^u := \varphi_i(I_e^u)$ for $u := 1, 2$. By default, let $I_1 := I_e^1$ be a new base interval, define y_{s_1} to be the middle point of J_e^1. If at a later stage $s_2 > s_1$, $\varphi_i(\gamma_j(s_2))$ enters the interval J_e^1, then set $I_1 := I_e^2$. If there is another $s_3 > s_2$ such that $\varphi_i(\gamma_j(s_2))$ enters the interval J_e^2, then redefine $I_1 := I_e^1$, and so on. In each case, we will define a new value of (y_s) as the middle point of I_1. Of course, this redefinition can appear at most $2^{(2r_1+1)}$ times if $\sum_{s\in\mathbb{N}} |\gamma_j(s) - \gamma_j(s+1)| \leq 1$.

Now on the base interval I_1 we will look for another minimal $e_1 := \langle i_1, j_1\rangle > e$ such that Lemma 6.1 can be applied to φ_{i_1}. Define r_2, $I_{e_1}^u$, $J_{e_1}^u$ $(u := 1, 2)$, I_2 and new y_s similarly. This procedure can be carried out further. By the above strategy, we can see that, first, the limit $y := \lim_{s\to\infty} y_s$ exists. In fact it is the unique common point of a nested interval sequence $(I_e)_{e\in\mathbb{N}}$; Second, every requirement $R_{\langle i,j\rangle}$ is satisfied, because $\lim_{s\to\infty} \varphi_i(\gamma_j(s))$ and y have at least the distance $|\varphi_i(a_2) - \varphi_i(a_3)|$ (for some $a_2 < a_3$), if φ_i and γ_j satisfy the premise of $R_{\langle i,j\rangle}$; Third, the n-divergence of (y_s) is bounded by a recursive function h defined by $h(n) := \sum_{m<n} 2^{2m+2}$. Here the third claim follows from the observation that we define new y_s only according to some requirement and some natural number r which satisfies Lemma 6.1 and any jump which is related to this r is not greater than $2^{-(r+1)}$. Different requirements relate to different such r and, for any fixed r, there are at most 2^{2r+2} jumps related to this r.

Unfortunately, the construction above is not effective, because, first, we cannot decide whether φ_i is a monotone total function and, second, we can't calculate the value $\varphi_i(\gamma_j(s))$ in finite steps, even if it is defined. To solve this problem, let $\beta_i :\subseteq [0;1]_{\mathbb{Q}} \times \mathbb{N} \to [0;1]_{\mathbb{Q}}$ be an approximation of φ_i such that $|\varphi_i(x) - \beta_i(x, n)| \leq 2^{-n}$ and use the function pair $(\beta_{i,s}, \gamma_{j,s})$ in the construction instead of (φ_i, γ_j). In this case, the finite injury priority method should be applied.

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