On the Jordan Decomposability for Computable Functions of Bounded Variation

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Abstract

According to Jordan Decomposition Theorem, every real function of bounded variation can be decomposed to a difference of two increasing functions. In this paper we will show, among others, that an effective version of this theorem does not hold for computable function of bounded variation.

Key words: Computable real function, bounded variation, Jordan decomposition

1 Introduction

Let $[a;b]$ be a real interval and $f : [a;b] \to \mathbb{R}$ a real function. The *variation of f over* $[a;b]$, denoted by $V_a^b(f)$, is defined by

$$V_a^b(f) := \sup \sum_{i<n} |f(a_i) - f(a_{i+1})| \tag{1}$$

where the supremum is taken over all possible subdivisions $a = a_0 < a_1 < \cdots < a_n = b$ of the interval $[0;1]$. If $V_a^b(f)$ is finite, then we say that f is of *bounded variation on* $[a;b]$ (BV on $[a;b]$ for short). The class of all BV-functions on $[a;b]$ is denoted by $\mathbb{BV}[a,b]$ or simply $\mathbb{BV}$ if the underlying interval is clear from the context. If f is a BV-function, then the function v_f defined by $v_f(x) := V_a^x(f)$ is called the *total variation function* of f. The concept of function of bounded variation is originated by Camille Jordan in [6]. The most important phenomenon for a BV-function is the Jordan decomposition.

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Namely, any BV-function f can be expressed as a difference of two increasing functions. In this paper the increasing function f means always that f is non-strictly increasing, i.e., $f(x) \leq f(y)$ for any $x \leq y$. More general properties of BV-functions and their applications are widely discussed in classical mathematics, effective analysis as well as in constructive mathematics ([1,3,5,8,10]).

In this paper, we are interested in the computable real functions $f : [a; b] \to \mathbb{R}$ which are of bounded variation (CBV-function, for short) for some computable real numbers $a < b$. First, we recall briefly the definition of computability on real numbers and real functions. We explain informally the approach of Weihrauch. For a more precise definition, the reader should refer to [12,13].

A sequence (x_s) of rational numbers is called computable if there is a computable function $f : \mathbb{N} \to \mathbb{Q}$ such that $x_s = f(s)$ for any $s \in \mathbb{N}$. For any real number x, a $\rho_<$-name ($\rho_>$-name) of x is an increasing (decreasing) sequence (x_s) of rational numbers converging to x. And a ρ-name of x is a sequence (x_s) of rational numbers which converges effectively (or fast) to x in the sense that $\forall s \in \mathbb{N}(|x - x_s| \leq 2^{-s})$. x is called *computable* (*left computable, or right computable*) if x has a computable ρ-name ($\rho_<$-, or $\rho_>$-name, respectively). In other words, a real number x is left (right) computable if there is an increasing (decreasing) computable sequence (x_s) of rational numbers which converges to x, while x is computable means that there is a computable sequence (x_s) of rational numbers converging effectively to x. A sequence (x_s) of real numbers is called computable if there is a computable sequence (r_{st}) of rational numbers which converges to (x_s) effectively and uniformly, i.e., $|x_s - r_{st}| \leq 2^{-t}$ for any $t, s \in \mathbb{N}$.

Let a, b be two computable real numbers such that $a < b$. For any $\delta_1, \delta_2 \in \{\rho, \rho_<, \rho_>\}$, a real function $f : [a; b] \to \mathbb{R}$ is called (δ_1, δ_2)-*computable* if there is an algorithm (or so-called type-2 Turing machine [1]) M which (δ_1, δ_2)-computes the function f in the sense that, for any $x \in [a; b]$ and any δ_1-name α of x, $M(\alpha)$ outputs a δ_2-name of $f(x)$. The computability of real functions with more than one arguments can be defined accordingly. Usually, the (ρ, ρ)-computable functions are simply called *computable* and $(\rho, \rho_<)$-computable functions are called *lower-semi-computable* (l.s.comp for short, see [14,16]). Computable and l.s.comp. real functions are effective counterparts of continuous and lower-semi-continuous functions, respectively. According to Effective Weierstrass Theorem (Lemma 6.1.10 of [13], see also page 45 of [9]), a function $f : [a; b] \to \mathbb{R}$ is computable iff there is a computable sequence (p_s) of rational polygon functions which converges effectively and uniformly to f, i.e., $\forall s \forall x \in [a; b] \, (|f(x) - p_s(x)| \leq 2^{-s})$.

Notice that the maximal value $\max\{f(x) : x \in [a; b]\}$ of a computable function

[1] The type-2 Turing machine extends the classical Turing machine in such a way that it accepts infinite sequence as well as finite string for inputs and outputs.

$f : [a; b] \to \mathbb{R}$ on a computable interval $[a; b]$ is computable (see e.g., Theorem 7, page 40 of [9]). Since $V_a^b(f) = V_0^1(g)$ for a function $g : [0; 1] \to [0; 1]$ defined by $g(x) := f(a + (b - a)x)/m$ where $m := \max\{f(x) : x \in [a; b]\}$, we can restrict ourselves w.l.o.g. only to the functions $f : [0; 1] \to [0; 1]$. The class of all CBV-functions $f : [0; 1] \to [0; 1]$ is denoted by $\mathbb{CBV}[0; 1]$ or simply $\mathbb{CBV}$.

Back to our main topic. The classic Jordan decomposition for BV-function does not hold for the Constructive Mathematics (CM) of Bishop and Bridges [2] as shown by Bridges [3]. In CM, the statement "a difference of any two increasing functions on $[0; 1]$ has a variation" entails the *Limited Principle of Omniscience* [2] (LPO) which cannot be derived in intuitionistic logic. Since the "existence" for CM means "constructible", this observation means roughly that "the difference of two computable increasing functions does not always have a computable variation". Bridges's argument bases on the Brouwerian example and hence does not lead to an concrete counterexample. Such an example will be constructed in this paper. That is, we will construct two increasing computable functions such that their difference does not have a computable variation. In fact, we obtain a stronger result that the variation $V_0^1(f)$ of any CBV-function f is left computable and every left computable real number is a variation of a CBV-function which is a difference of two increasing computable functions. Since there are left computable real numbers which are not computable by Specker [11], Bridges observation follows directly.

Because any computable real function maps a computable real number to a computable one. The above result implies that the total variation function v_f of a CBV-function f is not necessarily computable. Surprisingly, we show also that the total variation function v_f of an $f \in \mathbb{CBV}[0; 1]$ is a computable function iff the variation $V_0^1(f)$ of f is a computable real number. Namely, the computability of v_f as a function dependents only on the computability of one of its value $v_f(1)$ as a real number. While v_f (for $f \in \mathbb{CBV}$) is not necessarily computable, we can show that, it is always a lower-semi-computable function.

Another problem we will address in this paper is the possible effectivization of the Jordan decomposition. We call a function $f \in \mathbb{CBV}$ effective Jordan decomposable (EJD, for short), if there are two increasing computable functions f_1 and f_2 such that $f = f_1 - f_2$. Ko [7] shows that there is a polynomial time computable [3] function f of bounded variation which is not a difference

[2] The limited principle of omniscience says that, "if (a_n) is a binary sequence, then either there exists n such that $a_n = 1$ or else $a_n = 0$ for each n". For a comprehensive explanation about LPO please refer [4].

[3] Roughly, f is polynomial time computable, if there is a type-2 Turing machine M which computes the f in polynomial time. That is, there is a polynomial p such that, for any $x \in [0; 1]$ and any sequence (x_s) of rational numbers fast converging to x, M computes the n-th element y_n of an output sequence (y_s) converging effectively to $f(x)$ always in $p(n)$ steps.

of two increasing polynomial time computable functions. Namely, the polynomial time version of Jordan decomposition does not exist. In this paper, we will show that a general effective version of Jordan decomposition fails too. Namely, there is a computable function of bounded variation which is not effectively Jordan decomposable.

In section 2, we discuss the properties of the variation $V_0^1(f)$ of the CBV-function f and show that $V_0^1(f)$ takes all possible left computable real numbers for $f \in \mathbb{CBV}$. Section 3 discuss the total variation function and show that v_f is lower-semi-computable for any $f \in \mathbb{CBV}$ and v_f is computable iff $v_f(1)$ is computable. At last, we show that the effective Jordan decomposition does not hold in section 4.

2 The Variation of CBV-Functions on an Interval

In this section, we discuss some basic properties about the variation $V_0^1(f)$ of a CBV functions $f : [0; 1] \to [0; 1]$. First, we summarize some classic results about general BV-functions as follows.

Theorem 2.1 (cf. Berberian [1]) *Let $a, b \in \mathbb{R}$ be any real numbers with $a < b$ and $f : [a; b] \to \mathbb{R}$ a real function. Then the following hold.*

(1) If f is a (continuous) BV-function, then v_f is a (continuous) increasing function. And if f is increasing, the $v_f = f$.
(2) $V_a^b(f) = V_a^c(f) + V_c^b(f)$ for any $f \in \mathbb{BV}$ and $c \in [a; b]$.
(3) f is a (continuous) BV function iff there are two increasing (continuous) functions f_1 and f_2 such that $f = f_1 - f_2$.
(4) If $f, g \in \mathbb{BV}$, then $f + g$, $c \cdot f$, $|f|$, $f \cdot g \in \mathbb{BV}$, (for any $c \in \mathbb{R}$). If, in addition $\forall x \in [a; b](|g(x)| \geq c)$ holds for some $c > 0$, then $f/g \in \mathbb{BV}$ too.
(5) Let $L_a^b(f)$ denote the length of the graph of function f on the interval $[a; b]$, then $V_a^b(f) + (b - a) \geq L_a^b(f) \geq ((V_a^b(f))^2 + (b - a)^2)^{1/2}$. Therefore, $f \in \mathbb{BV}[a; b]$ iff the graph of f has a finite length on $[a; b]$.

Obviously, the items 1, 2, 4. and 5 of the Theorem 2.1 hold also for the computable functions of bounded variation accordingly. For the item 3, let's look at the classic proof at first. For any BV function f, the total variation function v_f and the function g defined by $g(x) := v_f(x) - f(x)$ are always increasing. Thus f can always be expressed as a difference $v_f - g$ of two increasing functions. This representation is called the *Jordan decomposition* of f. To transfer this proof into the case of $\mathbb{CBV}$, we have to prove that v_f is computable. Unfortunately, it is not the case as we will see later. In fact, the item 3 does not holds correspondingly for $\mathbb{CBV}$ as we will see in section 4.

The class $\mathbb{BV}$ is also not closed under composition. In [15], Wo-Sang Young gives a necessary and sufficient condition on a function $g : [0;1] \to [0;1]$ such that $f \circ g \in \mathbb{BV}$ (resp. $g \circ f \in \mathbb{BV}$) for all $f \in \mathbb{BV}$. Namely, $f \circ g \in \mathbb{BV}$ for all $f \in \mathbb{BV}$ iff there is a positive integer N such that, for any $[a;b] \subseteq [0;1]$, $g^{-1}([a;b])$ can be expressed as a union of N intervals; and $g \circ f \in \mathbb{BV}$ for all $f \in \mathbb{BV}$ iff g satisfies a Lipschitz condition on $[0;1]$, i.e., $\forall x, y \in [0;1](|f(x) - f(y)| \leq c \cdot |x - y|)$ for some constant c. Thus, it is easy to see that the class $\mathbb{CBV}$ is not closed under the composition too.

Another observation about $\mathbb{BV}$ is that, if $f : [0;1] \to [0;1]$ is a continuous BV-function, then its variation $V_0^1(f)$ can be calculated by considering only the subdivisions from a dense subset of $[0;1]$ instead of all real subdivisions of $[0;1]$. More precisely, for any $A \subseteq [0;1]$, we define the *variation of f on A* as $V_A(f) := \sup_A \sum_{i<m} |f(x_i) - f(x_{i+1})|$, where the supremum $\sup_A$ is taken over all subdivisions $0 = x_0 < x_1 < x_2 < \cdots x_m = 1$ with $x_i \in A$. f is *of bounded variation* on A (denoted by $f \in \mathbb{BV}_A$) if $V_A(f)$ is finite. As an example, we consider the dyadic rational number set $\mathbb{D} := \bigcup_{n\in\mathbb{N}} \mathbb{D}_n$, where $\mathbb{D}_n := \{m \cdot 2^{-n} : \ m \in \mathbb{N} \ \& \ m \leq 2^n\}$. Then $\mathbb{D}$ is a dense subset of $[0;1]$

Proposition 2.2 *If $f \in \mathbb{BV}[0;1]$ is continuous, then $V_x^y(f) = V_{\mathbb{D}\cap[x;y]}(f)$ for any $x, y \in [0;1]$.*

Proof. Let $f : [0;1] \to [0;1]$ be a continuous BV-function. Then f is uniformly continuous on the interval $[0;1]$. That is, there is a modulus $\alpha : \mathbb{N} \to \mathbb{N}$ such that $|u - v| \leq 2^{-\alpha(n)} \implies |f(u) - f(v)| \leq 2^{-n}$ for any $n \in \mathbb{N}$ and $u, v \in [0;1]$. Given any $x, y \in [0;1]$, the inequality $V_x^y(f) \geq V_{\mathbb{D}\cap[x;y]}(f)$ holds trivially by the definition of variation. It suffices now to show that $V_x^y(f) \leq V_{\mathbb{D}\cap[x;y]}(f)$ holds too.

By definition of $V_x^y(f)$, for any positive real number ε, there is a subdivision $x = x_0 < x_1 < \cdots < x_m = y$ of the interval $[x;y]$ such that $V_x^y(f) - \varepsilon/2 < \sum_{i<m} |f(x_i) - f(x_{i+1})|$. Fix an $n \in \mathbb{N}$ such that $m \cdot 2^{-n+1} < \varepsilon/2$ and choose, for all $i \leq m$, an $r_i \in \mathbb{D} \cap [x;y]$ such that $|x_i - r_i| \leq 2^{-\alpha(n)}$. Then, $\sum_{i<m} |f(r_i) - f(r_{i+1})| \geq \sum_{i<m}(|f(x_i) - f(x_{i+1})| - |f(r_i) - f(x_i)| - |f(x_{i+1}) - f(r_{i+1})|) \geq V_x^y(f) - \varepsilon/2 - 2m \cdot 2^{-n} \geq V_x^y(f) - \varepsilon$. This implies that $V_{\mathbb{D}\cap[x;y]}(f) = \sup_{\mathbb{D}\cap[x;y]} \sum_{i<m} |f(r_i) - f(r_{i+1})| \geq V_x^y(f)$. $\square$

By Proposition 2.2, the variation $V_0^1(f)$ can be calculated by considering all subdivisions of $[0;1]$ which consists of dyadic rational numbers instead of all real number subdivisions. This is especially useful if the function f is computable. In this case, we can approximate the variation $V_0^1(f)$ from below.

Theorem 2.3 *Let $f \in \mathbb{CBV}[0;1]$. Then the following hold.*

(1) $V_0^1(f)$ is a left computable real number.

(2) *For any left computable real number c, there is an effective Jordan decomposable function $f \in \mathbb{CBV}[0;1]$ such that $V_0^1(f) = c$.*

Proof. The item 1. follows directly from a more general result of Lemma 3.1 in section 3.

We prove now the item 2. Since $V_0^1(a \cdot f) = a \cdot V_0^1(f)$, for any $a \in \mathbb{R}$ and $f \in \mathbb{BV}$, by the definition, it suffices to consider the left computable real number $c \in [0;1]$. Let (c_s) be a computable increasing sequence of rational numbers which converges to c. Suppose without lost of generality that $c_0 = 0$ and $c_s < c_{s+1}$ for any $s \in \mathbb{N}$. We define a recursive function $k : \mathbb{N} \to \mathbb{N}$ by

$$k(n) := \mu m > 0 \left(\frac{c_{n+1} - c_n}{2m} < 2^{-n} \right)$$

for any $n \in \mathbb{N}$. Now the function $f : [0;1] \to [0;1]$ is defined as an infinite rational polygon function such that, on any interval $[c_s; c_{s+1}]$, for any $s \in \mathbb{N}$, f is a finite polygon function which connects all points

$$(c_s, 0), (c_s + e_s, e_s), (c_s + 2e_s, 0), \cdots, (c_s + (2k(s) - 1)e_s, e_s), (c_{s+1}, 0)$$

where $e_s := (c_{s+1} - c_s)/2k(s)$. In the interval $[c;1]$, define $f(x) := 0$ for any $x \in [c;1]$.

By the definition of f, $V_{c_s}^{c_{s+1}}(f) = 2k(s)e_s = (c_{s+1} - c_s)$. Therefore,

$$V_0^1(f) = \sum_{s=0}^{\infty} V_{c_s}^{c_{s+1}}(f) + V_c^1(f) = \sum_{s=0}^{\infty} (c_{s+1} - c_s) = c.$$

Thus f is of bounded variation on $[0;1]$.

To show that f is a computable function, let $p_s : [0;1 \to [0;1]$, for any $s \in \mathbb{N}$, be a rational polygon function defined by $p_s(x) := f(x)$ if $x \in [0; c_{s+1}]$ and $p_s(x) := 0$ otherwise. Then, (p_s) is a computable sequence of rational polygon functions. Because $|f(x) - p_s(x)| \leq e_s < 2^{-s}$ for any $x \in [0;1]$ and $s \in \mathbb{N}$, the sequence (p_s) converges effectively uniformly to f. Therefore, f is a computable function by Effective Weierstrass Theorem.

In order to show that $f = f_1 - f_2$ for two computable increasing functions f_1, f_2, define simply $f_1(x) := x$ and $f_2 := f_1 - f$. Then both f_1 and f_2 are computable. Obviously, f_1 is increasing. Since the function f has the increasing ratio either ± 1 or 0 on any subinterval of $[0;1]$ on which f is linear, it is not difficult to see that f_2 is increasing too. $\square$

3 The Total Variation Function of a CBV-Function

Last section discusses the variation of a CBV-function on a closed interval. In this section, we focus on the total variation function v_f of a CBV-function $f : [0;1] \to [0;1]$ and we will show that the v_f is not necessarily computable but lower-semi-computable. Furthermore, we show that v_f is computable iff $V_0^1(f)$ is computable. Since $v_f(1) = V_0^1(f)$, this means that the computability of the function v_f dependents in fact only on computability of one of its the value $v_f(1)$. Let's prove at first the following lemma.

Lemma 3.1 *For any $f \in \mathbb{CBV}[0;1]$, the function $w_f : [0;1]^2 \to \mathbb{R}$ defined by $w_f(x,y) := V_x^y(f)$ is a $(\rho_>, \rho_<, \rho_<)$-computable function.*

Proof. Let $f : [0;1] \to [0;1]$ be a computable function of bounded variation. By definition of the $(\rho_>, \rho_<, \rho_<)$-computability, it suffices to construct a type-2 Turing machine M such that, for any $x, y \in [0;1]$ and any $\rho_>$-name α of x and $\rho_<$-name β of y, $M(\alpha, \beta)$ outputs a $\rho_<$-name of $w_f(x,y)$.

Let (b_s) be an effective one-one enumeration of all dyadic rational numbers of $[0;1]$. Of course, (b_s) is also a computable sequence of real numbers. By sequential computability of the computable function f (see page 25 of [9]), the sequence $(f(b_s))$ is also a computable sequence of real numbers. That is, there is a computable double sequence (r_{st}) of rational numbers such that $|f(b_s) - r_{st}| \leq 2^{-t}$ for any $t, s \in \mathbb{N}$. Notice that, given any increasing sequence (x_s) of rational numbers converging to x, we can effectively construct an increasing sequence (x'_s) of dyadic rational numbers by $x'_s := \max\{r \in \mathbb{D}_s : r \leq x_s\}$ such that $x'_s \in \mathbb{D}_s$ and $\lim_{s\to\infty} x'_s = x$. For the decreasing sequence, it is similar. Therefore, we need only to consider the monotone sequence (x_s) of dyadic rational numbers with $x_s \in \mathbb{D}_s$ as inputs to the machine M.

The type-2 Turing machine M works as follows. Given a decreasing sequence (x_s) and an increasing sequence (y_s) of dyadic rational numbers such that $\forall s \in \mathbb{N}(x_s, y_s \in \mathbb{D}_s)$, $\lim_{s\to\infty} x_s = x$ and $\lim_{s\to\infty} y_s = y$, as inputs, M will output an increasing sequence (z_s) of rational numbers which converges to $w_f(x,y)$.

For any $s \in \mathbb{N}$, let $x_s = a_0 < a_1 < \cdots < a_k = y_s$ be the finest subdivision of the interval $[x_s; y_s]$ such that $a_i \in \mathbb{D}_s$ for all $i \leq k$. Then $V_{\mathbb{D}_s \cap [x_s;y_s]}(f) = \sum_{i<k} |f(a_i) - f(a_{i+1})|$. Define

$$u_s := \sum_{i<k} |f(a_i) - f(a_{i+1})| \quad \text{and} \quad v_s := \sum_{i<k} |e_i - e_{i+1}|$$

where $e_i := r_{t_i(2s+1)}$ for some $t_i \in \mathbb{N}$ such that $b_{t_i} = a_i$, i.e., a_i is the t_i-th element in the enumeration (b_n) of all dyadic rational numbers. Then $|f(a_i) - e_i| = |f(b_{t_i}) - r_{t_i(2s+1)}| \leq 2^{-(2s+1)}$ for all $i \leq k$. Obviously (u_t) is an increasing sequence such that $\lim_{t\to\infty} u_t = \lim_{s\to\infty} V_{\mathbb{D}_s\cap[x_s;y_s]}(f) = V_{\mathbb{D}\cap[x;y]}(f) = V_x^y(f)$. Furthermore, we have

$$\begin{aligned}|u_s - v_s| &\leq \sum_{i<k}(|f(a_i) - e_i| + |f(a_{i+1}) - e_{i+1}|)\\ &\leq \sum_{i<k} 2\cdot 2^{-(2s+1)} \leq 2^s \cdot 2 \cdot 2^{-(2s+1)} = 2^{-s}.\end{aligned}$$

This implies that $u_s - 2^{-s+1} \leq v_s - 2^{-s} \leq u_s$, for any $s \in \mathbb{N}$, and hence $u_s - 2^{-s+1} = \max_{t\leq s}(u_t - 2^{-t+1}) \leq \max_{t\leq s}(v_t - 2^{-t}) \leq \max_{t\leq s} u_t = u_s$. Now let $z_s := \max_{t\leq s}(v_t - 2^{-t})$. Then (z_s) is an increasing sequence of rational numbers such that $\lim_{s\to\infty} z_s = \lim_{s\to\infty} u_s = V_x^y(f)$. That is, it suffices to take (z_s) as the output of M. □

By Lemma 3.1, we can show the following theorem easily.

Theorem 3.2 *The total variation function v_f of any $f \in \mathbb{CBV}[0;1]$ is a continuous and increasing lower-semi-computable function.*

Proof. Let $f \in \mathbb{CBV}[0;1]$, the function w_f defined by $w_f(x,y) := V_x^y(f)$ is a $(\rho_>, \rho_<, \rho_<)$-computable function by the Lemma 3.1. Therefore, v_f is $(\rho_<, \rho_<)$-computable because $v_f(x) = w_f(0,x)$. Since any ρ-name of a real number can be effectively reduced to a $\rho_<$-name of the same real number, the $(\rho_<, \rho_<)$-computability implies $(\rho, \rho_<)$-computability. Therefore v_f is $(\rho, \rho_<)$-computable, i.e., lower-semi-computable. □

Since the lower-semi-computable functions are not necessarily computable (even not necessarily continuous), now we are going to investigate whether v_f is computable for $f \in \mathbb{CBV}$. Generally, the computability of a function $g : [0;1] \to [0;1]$ is a global property of g on the interval $[0;1]$. Although g maps any computable real number into a computable one if g is a computable function, there are also non-computable real function which can do that (see Theorem 6 of [9]). That is, the computability of $g(x)$ at some (or even all) computable x does not implies the computability of g in general. However, the situation for the total variation function v_f of a CBV-function f is different as shown in the next theorem.

Theorem 3.3 *For any $f \in \mathbb{CBV}[0,1]$, the total variation function v_f is computable iff the variation $V_0^1(f)$ is a computable real number.*

Proof. We prove the non-trivial direction. Suppose that $V_0^1(f)$ is a computable real number. There is a computable sequence (r_s) of rational numbers which converges to $V_0^1(f)$ effectively, i.e., $|V_0^1(f) - r_s| \leq 2^{-(s+1)}$ for any $s \in \mathbb{N}$. We will show that the total variation function v_f is (ρ, ρ)-computable.

By Lemma 3.1, there is a type-2 Turing machine M_1 which $(\rho_>, \rho_<, \rho_<)$-computes the function w_f, i.e., for any $\rho_>$-name α of x and any $\rho_<$-name β of y, $M_1(\alpha, \beta)$ outputs a $\rho_<$-name of $w_f(x, y) := V_x^y(f)$. We construct a type-2 Turing machine M which (ρ, ρ)-computes the function v_f as follows. For any $x \in [0; 1]$ and any ρ-name (x_s) of x, let $a_s := x_s - 2^{-s}$ and $b_s := x_s + 2^{-s}$ for any $s \in \mathbb{N}$. Then (a_s) and (b_s) are increasing and decreasing, respectively, sequences of rational numbers which converge to x. Let $c_s := 0$ and $d_s = 1$ for any $s \in \mathbb{N}$. By simulating the type-2 Turing machine M_1 on the inputs $((c_s), (a_s))$ and $((b_s), (d_s))$, we get two increasing sequences (u_s) and (v_s) of rational numbers which converge to $V_0^x(f)$ and $V_x^1(f)$ respectively. Since $\lim_{t\to\infty}(u_t + v_t) = V_0^1(f) = \lim_{t\to\infty} r_t$, we can effectively choose a $t \in \mathbb{N}$ such that $|(u_t + v_t) - r_{s+1}| \leq 2^{-(s+1)}$ and define $y_s := u_t$, for any $s \in \mathbb{N}$. Then $|v_f(x) - y_s| = (V_0^x(f) - u_t) \leq (V_0^x(f) - u_t) + (V_x^1(f) - v_t) = V_0^1(f) - (u_t + v_t) \leq |V_0^1(f) - r_{s+1}| + |r_{s+1} - (u_t + v_t)| \leq 2^{-s}$. That is, the sequence (y_s) converges to $v_f(x)$ effectively and hence it is a ρ-name of $v_f(x)$.

Therefore, M does (ρ, ρ)-compute the function v_f and v_f is a computable function. □

By Theorem 2.3.2 and Theorem 3.3, we have the following corollary.

Corollary 3.4 *There is a computable function f of bounded variation whose total variation function v_f is not computable.*

Another direct outcome of the Theorem 3.3 is that the converse of the Theorem 3.2 does not hold. That is, there is a continuous and increasing lower-semi-computable function v which is not a total variation function of any CBV-function. For example, let $b \in (1/2; 1)$ be a left computable but not computable real number and v a polygon function which connects the points $(0, 0), (1/2, b)$ and $(1, 1)$. Then, v is lower-semi-computable but not computable function because $v(1/2) = b$ is not computable while $v(1) = 1$ is a computable real number. By Theorem 3.3, v cannot be a total variation function of any $f \in \mathbb{CBV}$.

4 Effective Jordan Decomposability

In this section, we will discuss the effective Jordan decomposability of a CBV-function. We investigate the condition under which a CBV function is EJD and we will show at last that not every CBV function is EJD.

First of all, we have an easy sufficient but not necessary condition of EJD as follows.

Lemma 4.1 *(1) If the total variation function v_f is computable, then f is EJD.*

(2) There is a CBV-function $f : [0;1] \to [0;1]$ which is EJD but its total variation function v_f is not computable

Proof. 1. It suffices to consider the decomposition of $f = v_f - (v_f - f)$.

2. By the Theorem 2.3.2. there is an EJD function $f \in \mathbb{CBV}[0;1]$ such that the variation $V_0^1(f)$ is left computable but not computable. By the Theorem 3.3, the total variation function v_f is not computable. □

The next lemma gives a necessary condition of EJD.

Lemma 4.2 *If f is an EJD function, then the total variation function v_f has a computable modulus of continuity.*

Proof. Let $f = f_1 - f_2$ and f_1, f_2 are increasing computable functions. Let α_1 and α_2 are computable modulus of continuity of f_1 and f_2, respectively. Since $|v_f(y) - v_f(x)| = V_x^y(f) = V_x^y(f_1 - f_2) \leq (f_1(y) - f_1(x)) + (f_2(y) - f_2(x))$, the computable function α defined by $\alpha(n) := \max\{\alpha_1(n+1), \alpha_2(n+1)\}$ is a modulus of continuity of f. □

Unfortunately, we have not found a necessary and sufficient condition for EJD functions. Nevertheless, the next theorem shows that not every CBV-function is EJD.

Theorem 4.3 *There is a function $f \in \mathbb{CBV}[0;1]$ which is not EJD.*

Proof. Let $A \subseteq \mathbb{N}$ be a non-recursive r.e. set and $\{n_0, n_1, n_2, \cdots\}$ an effective one-one enumeration of A. Then $x_A := \sum_{i\in\mathbb{N}} 2^{-i}$ is a left computable but not computable real number. Denote by A_s the finite set $\{n_0, n_1, \cdots n_s\}$ and $a_s := \sum_{i\in A_s} 2^{-i}$. Then $\lim_{s\to\infty} a_s = x_A$. Let $m_s := \max\{n_t + s + 1 : t \leq s\}$ and $t_s := m_s - (n_s + 1)$ for any $s \in \mathbb{N}$. Then we have $t_s \geq 0$ and $m_s < m_{s+1}$ for any $s \in \mathbb{N}$. Now the function $f : [0;1] \to [0;1]$ is defined as an infinite rational polygon by

$$f(x) := \begin{cases} 2^{-m_s} \cdot \delta(2^{t_s+s+1}(x - 2^{-(s+1)})) & \text{if } x \in [2^{-(s+1)}; 2^{-s}] \\ 0 & \text{if } x = 0. \end{cases} \tag{2}$$

where $\delta : \mathbb{R} \to \mathbb{R}$ is a basic zigzag function defined for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$ by

$$\delta(x) := \begin{cases} 2(x-n) & \text{if } x \in [n; n+1/2) \\ 2-2(x-n) & \text{if } x \in [n+1/2; n+1]. \end{cases} \tag{3}$$

We will show that $f \in \mathbb{CBV}[0;1]$ but v_f does not have a computable modulus of continuity, hence is not EJD by Lemma 4.2.

First we show that f is computable. To this end, we define a computable sequence (p_n) of rational polygon functions by

$$p_n(x) := \begin{cases} f(x) & \text{if } x \in [2^{-(n+1)}; 1] \\ 0 & \text{if } x \in [0; 2^{-(n+1)}). \end{cases}$$

Since $\max\{|f(x)-p_s(x)| : x \in [0;1]\} = \max\{|f(x)-p_s(x)| : x \in [0; 2^{-(s+1)}]\} \leq 2^{-m_s} \leq 2^{-s}$, the computable sequence (p_s) converges uniformly effectively to the function f. Therefore f is computable by Effective Weierstrass Theorem.

Second, by the definition of the function f, it is easy to see that $V_0^1(f) = \sum_{s\in\mathbb{N}} V_{2^{-(s+1)}}^{2^{-s}}(f) = \sum_{s\in\mathbb{N}} 2 \cdot 2^{t_s} \cdot 2^{-m_s} = \sum_{s\in\mathbb{N}} 2^{-n_s} = x_A$. Therefore, f is of bounded variation on $[0;1]$, i.e., $f \in \mathbb{CBV}[0;1]$.

Finally, we show that the function v_f does not have a computable modulus of continuity. Assume by contradiction that v_f has a computable modulus of continuity $\alpha : \mathbb{N} \to \mathbb{N}$. That is, for any $x, y \in [0;1]$ and $s \in \mathbb{N}$, $|v_f(x) - v_f(y)| \leq 2^{-s}$ holds if $|x-y| \leq 2^{-\alpha(s)}$. Especially, for any $s \in \mathbb{N}$, we have

$$\sum_{t \geq m(s)} 2^{-n_t} = V_0^{2^{-\alpha(s)}}(f) = |v_f(0) - v_f(2^{-\alpha(s)})| \leq 2^{-s}.$$

This implies that $(\forall t)(\forall s)(t \geq \alpha(s) \implies n_t < s)$ and hence, for any $n \in \mathbb{N}$, $n \in A \iff n \in A_{\alpha(n)}$. Therefore, A is a recursive set because α is a computable function. This contradicts the hypothesis. □

References

[1] S. K. Berberian. *Fundamentals of Real Analysis.* Universitext. Springer, New York, Berlin, Heidelberg, 1998.

[2] E. Bishop and D. S. Bridges. *Constructive Analysis*, volume 279 of *Grundlehren der mathematischen Wissenschaften.* Springer, Berlin, 1985.

[3] D. Bridges. A constructive look at functions of bounded variation. *Bull. London Math. Soc.*, 32(3):316–324, 2000.

[4] D. Bridges and F. Richman. *Varieties of Constructive Mathematics*, volume 97 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1987.

[5] E. Casas, K. Kunisch, and C. Pola. Some applications of bv functions in optimal control and calculus of variations. *ESAIM: Proceedings*, 4:83–96, 1998.

[6] C. Jordan. *Cours d'analyse de l'Ecole Polytechnique*. 1982–1887.

[7] K.-I. Ko. Some negative results on the computational complexity of total variation and differentiation. *Inform. Contr.*, 53:21–31, 1982.

[8] K.-I. Ko. *Complexity Theory of Real Functions*. Progress in Theoretical Computer Science. Birkhäuser, Boston, 1991.

[9] M. B. Pour-El and J. I. Richards. *Computability in Analysis and Physics*. Perspectives in Mathematical Logic. Springer, Berlin, 1989.

[10] R. Rettinger, X. Zheng, R. Gengler, and B. von Braunmühl. Weakly computable real numbers and total computable real functions. In *Proceedings of COCOON 2001, Guilin, China, August 20-23, 2001*, volume 2108 of *LNCS*, pages 586–595. Springer, 2001.

[11] E. Specker. Nicht konstruktiv beweisbare Sätze der Analysis. *The Journal of Symbolic Logic*, 14(3):145–158, 1949.

[12] K. Weihrauch. *Computability*, volume 9 of *EATCS Monographs on Theoretical Computer Science*. Springer, Berlin, 1987.

[13] K. Weihrauch. *Computable Analysis, An Introduction.* Springer, Berlin Heidelberg, 2000.

[14] K. Weihrauch and X. Zheng. Computability on continuous, lower semi-continuous and upper semi-continuous real functions. *Theoretical Computer Science*, 234:109–133, 2000.

[15] W.-S. Young. Composing functions of bounded variation. *Proc. Amer. Math. Soc.*, 83:254–356, 1981.

[16] X. Zheng, V. Brattka, and K. Weihrauch. Approaches to effective semi-continuity of real functions. *Mathematical Logic Quarterly*, 45(4):481–496, 1999.