Computable Functions of Bounded Variation and Semi-Computable Real Numbers

Robert Rettinger [a], Xizhong Zheng [b,*], Burchard von Braunmühl [b]

[a] *FernUniversität Hagen, 58084 Hagen, Germany*

[b] *BTU Cottbus, 03044 Cottbus, Germany*

Abstract

In this paper we discuss some basic properties of computable real functions which have bounded variations (CBV-functions for short). Especially, it is shown that the image set of semi-computable real numbers under CBV-functions is a proper subset of weakly computable real number class; Two applications of CBV-functions to semi-computable real numbers produce the whole closure of semi-computable real numbers under total computable real functions, and the image sets of semi-computable real numbers under monotone computable functions and CBV-functions are different.

Key words: Computable real function, Bounded variation, Weakly computable real number,

1 Introduction

The continuity of a real function is the most important property in analysis. The effective counterpart of a continuous real function is the computable real function which can be computed by a generalized Turing machine (Type-2 Turing machine, see [12,14]) based on the fast convergent Cauchy representation of real numbers. Computable real functions have been widely discussed in effective analysis (see e.g., [4,6,5,7,14]). However, there are many problems, especially in the applications to physical science we require more precise information about a function than the continuity or computability. For example, it is very useful to be able to measure how rapidly a real function f oscillates

* Corresponding author. email: zheng@informatik.tu-cottbus.de

on some interval $[a;b]$. But the oscillatory character of a function is not easily determined from its continuity or even its computability. For this reason, the notion of variation of a function is introduced in mathematics, actually by Camille Jordan (1838–1922). Precisely, the variation $V(f)[a;b]$ of a function f on the interval $[a;b]$ is defined as the supremum $\sup \sum_{i<k} |f(x_i) - f(x_{i+1})|$ which is taken over all possible subdivision $a = x_0 < x_1 < x_2 < \cdots < x_k = b$ of the interval. This quantity turns out to be very useful for problems in physics, engineering, probability theory, Fourier series, and so forth.

A function f is called of *bounded variation* on an interval $[a;b]$, if $V(f)[a;b]$ is finite (cf. [9]). Denote by $\mathbb{BV}[a;b]$ ($\mathbb{CBV}[a;b]$) the class of all (computable) real functions $f : [a;b] \to [0;1]$ which are of bounded variation on $[a;b]$. Especially, $\mathbb{BV}[0;1]$ and $\mathbb{CBV}[0;1]$ are denoted simply by $\mathbb{BV}$ and $\mathbb{CBV}$, respectively. The class $\mathbb{BV}$ is widely discussed in classical mathematics. In this paper, we are more interested in the class of $\mathbb{CBV}$. We will investigate, which classical properties of continuous function $f \in \mathbb{BV}$ can be extended to that of computable functions $f \in \mathbb{CBV}$. For example, let $v_f(x) := V(f)[0;x]$ for any $x \in [0;1]$, then the function v_f is continuous if f and its differential f' are both continuous. Accordingly, we can show that v_f is computable if both f and f' are computable. On the other hand, we show that $V(f)[0;1]$, if it exists, is not computable in general but left computable for a computable function f , and so on.

Moreover, we will discuss the closure property of several real number classes under the functions $\mathbb{CBV}$. Let $\mathbb{F}(\mathbf{C}) := \{f(x) : f \in \mathbb{F} \ \& \ x \in \mathbf{C}\}$ be the image set of the real number set $\mathbf{C} \subseteq \mathbb{R}$ under the function class $\mathbb{F}$. We will consider the relationship between $\mathbb{CBV}(\mathbf{C})$ and $\mathbf{C}$ or other classes of real numbers. In effective analysis, several interesting classes of real numbers are introduced and investigated. For example, the classes of computable, left computable, semi-computable, weakly computable and recursively approximable real numbers (see, [1,3,10,15,17]). Where a real number x is called *computable* if there is a computable sequence $(x_s)_{s\in\mathbb{N}}$ of rational numbers which converges effectively to x in the sense that $|x - x_s| \leq 2^{-s}$ for any $s \in \mathbb{N}$; x is *left (right) computable* if there is an increasing (decreasing) computable sequence of rational numbers which converges to x; Left and right computable real numbers are called *semi-computable*; x is *weakly computable* if there are left computable real numbers y and z such that $x = y - z$, and x is *recursively approximable* if it is a limit of a computable sequences of rational numbers. We denote by $\mathbf{EC}, \mathbf{LC}, \mathbf{RC}, \mathbf{SC}, \mathbf{WC}$ and $\mathbf{RA}$ the classes of computable, left computable, right computable, semi-computable, weakly computable and recursively approximable real numbers, respectively.

Obviously, the classes $\mathbf{EC}$ and $\mathbf{RA}$ are closed under the application of functions of $\mathbb{CBV}$. That is, if $f \in \mathbb{CBV}$ and $x \in \mathbf{EC}\,(\mathbf{RA})$, then $f(x) \in \mathbf{EC}\,(\mathbf{RA})$. Furthermore, let f be the computable function defined by $f(x) := 1 - x$ for any $x \in [0;1]$. Then f is a CBV-function which maps any left computable real number to a right one and vice versa. Moreover, $g \circ f \in \mathbb{CBV}$ if and only if $g \in \mathbb{CBV}$ for any computable function g. This observation implies that $\mathbb{CBV}(\mathbf{LC}) = \mathbb{CBV}(\mathbf{RC}) = \mathbb{CBV}(\mathbf{SC})$. Similarly, for the class $\mathbb{CTF}$ of all computable real functions $f : [0;1] \to [0;1]$, we have $\mathbb{CTF}(\mathbf{LC}) = \mathbb{CTF}(\mathbf{RC}) = \mathbb{CTF}(\mathbf{SC})$.

In [8], it is shown that both the classes **SC** and **WC** are not closed under $\mathbb{CBV}$ and that the image of a semi-computable real number under a CBV-function is weakly computable. In other words, $\mathbf{SC} \subsetneq \mathbb{CBV}(\mathbf{SC}) \subseteq \mathbf{WC}$ and $\mathbf{WC} \subsetneq \mathbb{CBV}(\mathbf{WC})$ hold. On the other hand, it is also shown in [8] that $\mathbf{WC} \subsetneq \mathbb{CTF}(\mathbf{SC}) = \mathbb{CTF}(\mathbf{WC}) \subsetneq \mathbf{RA}$. Namely, the image sets of semi-computable and weakly computable real numbers under total computable real functions are the same and they locate strictly between the classes **WC** and **RA**. Two interesting questions remain open there. Namely, whether $\mathbb{CBV}(\mathbf{SC}) = \mathbf{WC}$? and $\mathbb{CBV}(\mathbf{WC}) = \mathbb{CTF}(\mathbf{WC})$? For the first question, we will show a negative answer. That is, we show that $\mathbb{CBV}(\mathbf{SC}) \subsetneq \mathbf{WC}$. A positive answer for the second question follows from a more strong results that $\mathbb{CBV}^2(\mathbf{SC}) = \mathbb{CTF}(\mathbf{SC})$. This result shows that any application of a total computable real function to a semi-computable real number can be realized by two consecutive applications of the $\mathbb{CBV}$ functions to the some (possibly different) semi-computable real number.

In next section, we remind some related notions and known results. Section 3 shows some basic properties of CBV-functions. In section 4, we show that $\mathbb{CBV}(\mathbf{SC}) \subsetneq \mathbf{WC}$ and the proof of $\mathbb{CBV}^2(\mathbf{SC}) = \mathbb{CTF}(\mathbf{SC})$ is given in the last section 6.

2 Preliminaries

In this section we will recall some known results related to our new results which will be proved in later sections. Some notions and notations will also be explained here.

Let Σ be any alphabet. Σ^* and Σ^ω is the sets of all finite strings and infinite sequences of Σ, respectively. The set of all strings $w \in \Sigma^*$ of length n is denoted by Σ^n. For $u, v \in \Sigma^*$, denote by uv the concatenation of v after u. If $w \in \Sigma^* \cup \Sigma^\omega$, then $w[n]$ denotes its n-th element. Thus, $w = w[0]w[1] \cdots w[n-1]$, if $|w|$, the length of w, is n, and $w = w[0]w[1]w[2] \cdots$, if $|w| = \infty$. Obviously, $w[n]$ is defined only for $n < |w|$. We will say also that $w[n]$ is undefined and denote by $w[n] =\uparrow$, if $n \geq |w|$. The unique string of length 0 is denoted by λ (so-called empty string). For any finite string $w \in \{0;1\}^*$, and number $n \leq |w|$, the restriction $w \upharpoonright n$ is defined by $(w \upharpoonright n)[i] := w[i]$ is $i < n$ and

$(w \upharpoonright n)[i] := \uparrow$, otherwise. Then the length $|w \upharpoonright n| = n$.

We denote by $\mathbb{N}, \mathbb{Q}$ and $\mathbb{R}$ the sets of all natural, rational and real numbers, respectively. Let $\mathbb{D}_n := \{m \cdot 2^{-n} : m \leq 2^n\}$ be the set of all dyadic numbers of precision n and $\mathbb{D} := \bigcup_{n\in\mathbb{N}} \mathbb{D}_n$ is the set of all dyadic numbers both from the interval $[0; 1]$. $[0; 1]_\mathbb{Q}$ is the set of all rational numbers $x \in [0; 1]$. For any set A and B, $f :\subseteq A \to B$ is a partial function such that $\mathrm{dom}(f) \subseteq A$ and $\mathrm{range}(f) \subseteq B$, while $f : A \to B$ is a total function from A to B with $\mathrm{dom}(f) = A$ and $\mathrm{range}(f) \subseteq B$. For any interval $I \subset \mathbb{R}$, we denote its length by $l(I)$.

The computability notions about subsets of $\mathbb{N}$ and number-theoretical functions $f :\subseteq \mathbb{N}^k \to \mathbb{N}$ are well defined and developed in the classical computability theory (see e.g., [11,13]). Let $(\varphi_e)_{e\in\mathbb{N}}$ be an effective enumeration of all computable functions $\varphi_e : \mathbb{N} \to \mathbb{N}$. We can define φ_e as, say, the function computed by the e-th Turing machine M_e in some effective enumeration $(M_e)_{e\in\mathbb{N}}$ of all Turing machines. We write $\varphi_e(n) \downarrow$ if $n \in \mathrm{dom}(\varphi_e)$ and $\varphi_e(n) \uparrow$, otherwise. Then we can define an uniformly effective approximation $\varphi_{e,s}$ of φ_e by $\varphi_{e,s}(n) := m$, if $M_e(n)$ halts in s steps and outputs m; and $\varphi_{e,s}(n)$ is undefined, otherwise. Naturally, we assume that $e, n, m \leq s$ if $\varphi_{e,s}(n) = m$.

For any other countable set like $\mathbb{Q}$, the corresponding notions of computability can be defined accordingly by means of effective coding. For example, let $\langle \cdot, \cdot \rangle : \mathbb{N}^2 \to \mathbb{N}$ be a computable pairing function defined by $\langle n, m\rangle := (n + m)(n + m + 1)/2 + m$. Its inverse functions are denoted by π_1 and π_2 which are obviously also computable. It is easy to see that, $\pi_i(n) \leq n$ for any n and $i = 1, 2$. By this pairing function, we can define a coding $\sigma : \mathbb{N} \to \mathbb{Q}$ of rational numbers by $\sigma(\langle n, m\rangle) := n/(m + 1)$ for any $n, m \in \mathbb{N}$ and call a set $A \subseteq \mathbb{Q}$ recursive or recursively enumerable if the set $\sigma^{-1}(A) := \{n \in \mathbb{N} : \sigma(n) \in A\}$ is recursive or recursively enumerable, respectively. A function $f :\subseteq \mathbb{N} \to \mathbb{Q}$ is computable if there is a computable function $g :\subseteq \mathbb{N} \to \mathbb{N}$ such that $f(n) = \sigma \circ g(n)$ for any $n \in \mathrm{dom}(\sigma \circ g)$, and so forth. Other type of computable function which involves rational numbers can be defined accordingly. For each type of computable functions, we can define also an effective enumeration and their uniformly effective approximation. Besides, we call a sequence $(x_s)_{s\in\mathbb{N}}$ of rational numbers *computable* if there is a computable function $f : \mathbb{N} \to \mathbb{Q}$ such that $x_s = f(s)$ for any s.

The computability of real functions can be defined by Type-2 Turing machines of Weihrauch [12,14]. Where a type-2 Turing machine M extends the classical Turing machine in such a way that it accepts infinite sequences as well as finite strings as inputs and outputs. For any input p, $M(p)$ outputs a finite string q if $M(p)$ halts in finite steps with q in its write-only output tape, while $M(p)$ outputs an infinite sequence q means that $M(p)$ will never halt and keep writing the sequence q on its output tape. A real function $f :\subseteq \mathbb{R} \to \mathbb{R}$

is *computable*, if there is a type-2 Turing machine M which computes f in the sense that, for any $x \in \mathrm{dom}(f)$ and any sequence p (of rational numbers) which converges effectively to x, $M(p)$ outputs a sequence q (of rational numbers) which converges effectively to $f(x)$. From this definition, it is easy to see that any computable function is continuous at any point in its domain.

We have mentioned several classes of (weaker) computable real numbers in the last section. It is easy to see that, for any class $\mathbf{C}$ introduced there, we have $x \in \mathbf{C} \iff x \pm n \in \mathbf{C}$ for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$. In other words, x and $x \pm n$ have always the same kind of effectiveness mentioned above. Therefore, w.l.o.g., we assume that any real number discussed in this paper is in the interval $[0;1]$ except for being pointed out explicitly otherwise. By the similar reason, we consider also only the real functions $f : [0;1] \to [0;1]$ in this paper.

We summarize some important properties of real number classes mentioned in the last section as follows.

Theorem 2.1 (Weihrauch and Zheng [15])

(1) The classes $\mathbf{EC}, \mathbf{LC}, \mathbf{RC}, \mathbf{SC}, \mathbf{WC}$ *and* $\mathbf{RA}$ *are all different and have the following relationships*

$$\mathbf{EC} = \mathbf{LC} \cap \mathbf{RC} \subsetneq\subsetneq \mathbf{SC} = \mathbf{LC} \cup \mathbf{RC} \subsetneq \mathbf{WC} \subsetneq \mathbf{RA};$$

(2) $x \in \mathbf{LC}$ *if and only if* $-x \in \mathbf{RC}$*;*
(3) $x \in \mathbf{WC}$ *if and only if there is a computable sequence* $(x_s)_{s\in\mathbb{N}}$ *of rational numbers which converges to* x *such that* $\sum_{s\in\mathbb{N}} |x_s - x_{s+1}| \leq 1$*; and*
(4) The classes $\mathbf{EC}, \mathbf{WC}$ *and* $\mathbf{RA}$ *are algebraic fields. That is, they are closed under the arithmetic operations* $+, -, \times$ *and* $\div$*.*

The closure properties of the above classes under partial computable real functions are first discussed in [16]. It is shown there that all classes except $\mathbf{EC}$ and $\mathbf{RA}$ are not closed under partial computable real functions. These results have been extended to the cases of $\mathbb{CTF}$ and $\mathbb{CBV}$ in [8] where the divergence bounded computability is also introduced.

Definition 2.2 (Rettinger et al. [8]) For any sequence (x_s) and $n \in \mathbb{N}$, the n-divergence of (x_s) is defined as the maximal m such that, for some chain $i_0 < j_0 \leq i_1 < j_1 \leq \cdots \leq i_m < j_m$ of natural numbers, $|x_{i_s} - x_{j_s}| \geq 2^{-n}$ holds for any $s \leq m$. A real number x is called *divergence bounded computable* if there is a recursive function h and a computable sequence (x_s) of rational numbers converging to x such that the n-divergence of (x_s) is bounded by $h(n)$ for any n. The class of all divergence bounded computable real numbers is denoted by $\mathbf{DBC}$.

We summarize now some known results in next theorem.

Theorem 2.3 (Zheng [16] and Rettinger et al. [8])

(1) $\mathbb{CPF}(\mathbf{LC}) = \mathbb{CPF}(\mathbf{RC}) = \mathbb{CPF}(\mathbf{RA}) = \mathbf{RA}$*;*
(2) $\mathbb{CTF}(\mathbf{LC}) = \mathbb{CTF}(\mathbf{RC}) = \mathbb{CTF}(\mathbf{WC}) = \mathbf{DBC}$*;*
(3) $\mathbf{SC} \subsetneq \mathbb{CBV}(\mathbf{LC}) = \mathbb{CBV}(\mathbf{RC}) = \mathbb{CBV}(\mathbf{SC}) \subseteq \mathbf{WC}$*;*
(4) $\mathbf{WC} \subsetneq \mathbb{CBV}(\mathbf{WC}) \subseteq \mathbb{CTF}(\mathbf{WC}) \subsetneq \mathbf{RA}$.

Since the image sets of three classes $\mathbf{LC}, \mathbf{RC}$ and $\mathbf{SC}$ under the functions classes $\mathbb{CPF}, \mathbb{CTF}$ and $\mathbb{CBV}$ are the same respectively, we need will state our results about only one of the equivalent classes later without further explanation.

Next, we introduce a notion of nested interval tree which will be used to construct some special functions.

Definition 2.4 (Rettinger et al. [8]) Let $\delta : \mathbb{N} \to \mathbb{N}^+$, $\mathbb{I}$ the set of all rational intervals on $[0;1]$ and $\mathbb{N}^*_\delta := \{w \in \mathbb{N}^* : \forall n < |w| \, (w[n] < \delta(n))\}$.

1. A *δ-interval tree* (δ-i.t., for short) on $[0;1]$ is a function $I : \mathbb{N}^*_\delta \to \mathbb{I}$ such that $I(\lambda) = [0;1]$; $\forall w \in \mathbb{N}^*_\delta(\bigcup_{i<\delta(|w|)} I(wi) = I(w))$, and $\lim_{|w_s|\to\infty} l(I(w_s)) = 0$. For any δ-i.t. I and $w \in \mathbb{N}^*_\delta$, the length, left and right endpoints of $I(w)$ are denoted by l^δ_w, a^δ_w and b^δ_w, respectively.

2. A δ-i.t. I is called *computable* if there are computable functions $a, b : \mathbb{N}^*_\delta \to [0;1]_\mathbb{Q}$ such that $a(w) := a^\delta_w$ and $b(w) := b^\delta_w$ for any $w \in \mathbb{N}^*_\delta$.

3. A δ-i.t. I is called *canonical* if, for any $w \in \mathbb{N}^*_\delta$, the interval $I(w)$ is divided into subintervals $I(w0), I(w1), \cdots, I(w(\delta(|w|) - 1))$ disjunctively and equally, in other words,

$$a^\delta_w := \sum_{i<|w|} \left(w[i] \cdot \prod_{j \le i} \delta(j)^{-1} \right) \text{ and } b^\delta_w := a^\delta_w + \prod_{j<|w|} \delta(j)^{-1} \tag{1}$$

where we make the convention that $\sum_{i\in\emptyset}(\cdots) := 0$ and $\prod_{i\in\emptyset}(\cdots) := 1$.

Using the δ_1-i.t I_1 and δ_2-i.t I_2 on x-axis and y-axis, respectively, we can define a function f which maps the interval $I_1(w)$ into the interval $I_2(\iota(w))$ if $\iota :\subseteq \mathbb{N}^*_{\delta_1} \to \mathbb{N}^*_{\delta_2}$ is compatible in the following sense.

Definition 2.5 (Rettinger et al. [8]) Let $\delta_1, \delta_2 : \mathbb{N} \to \mathbb{N}^+$. A function $\iota :\subseteq \mathbb{N}^*_{\delta_1} \to \mathbb{N}^*_{\delta_2}$ is called (δ_1, δ_2)-*compatible* if the following conditions hold.

1. The domain $\mathrm{dom}(\iota)$ of ι is infinite;

2. $\mathrm{dom}(\iota)$ is *alternate* in the sense that, for any $w \in \mathbb{N}^*_{\delta_1}$ and $i < \delta_1(|w|)$

$$\forall w \forall i < \delta_1(|w|)(wi \in \mathrm{dom}(\iota) \Longrightarrow w(i-1), w(i+1) \notin \mathrm{dom}(\iota) \ \&\ i \neq 0, \delta_1);$$

3. $\forall w, v \in \mathbb{N}^*_{\delta_1} \,(w \in \mathrm{dom}(\iota) \ \&\ v \sqsubseteq w \Longrightarrow v \in \mathrm{dom}(\iota) \ \&\ \iota(v) \sqsubseteq \iota(w)))$; and

4. $\forall u, v \in \mathrm{dom}(\iota) \,(|u| = |v| \Longrightarrow |\iota(u)| = |\iota(v)|)$.

The most important application of these notions is the following technical lemma which is very useful to construct some computable real functions.

Lemma 2.6 (Rettinger et al. [8]) *Let $\delta_1, \delta_2, e : \mathbb{N} \to \mathbb{N}^+$ be computable functions, I_1 a canonical δ_1-i.t., and I_2 a computable δ_2-i.t. such that $l^{\delta_2}_w \leq 2^{-e(|w|)}$ all $w \in \mathbb{N}^*_{\delta_2}$. If $\iota :\subseteq \mathbb{N}^*_{\delta_1} \to \mathbb{N}^*_{\delta_2}$ is a (δ_1, δ_2)-compatible computable function, then there is a computable function $f : [0;1] \to [0;1]$ such that $f(I_1(w)) \subseteq I_2(\iota(w))$ and $f(a^{\delta_1}_w) = a^{\delta_2}_{\iota(w)}$ for all $w \in \mathrm{dom}(\iota)$.*

3 Computable Function of Bounded Variation

In this section, we will discuss some basic properties of computable real functions of bounded variation on the interval $[0; 1]$. We investigate especially which properties of continuous functions of $\mathbb{BV}$ can be extended to that of computable real functions accordingly. For the $\mathbb{BV}$ functions, we have at first the following simple properties which hold obviously for the $\mathbb{CBV}$ functions too.

Proposition 3.1 *1. If $f, g \in \mathbb{BV}$, then $f - g, f \cdot g, f + g \in \mathbb{BV}$. If, in addition, $\exists c > 0 \forall x \in [0; 1](|g(x)| \geq c)$ holds, then $f/g \in \mathbb{BV}$;*

2. Let $L(f)[a; b]$ be the length of the graph of function f on $[a; b]$, then

$$V(f)[a; b] + (b - a) \geq L(f)[a; b] \geq [(V(f)[a; b])^2 + (b - a)^2]^{1/2}.$$

Therefore, $f \in \mathbb{CBV}[0; 1]$ if and only if the graph of f on $[a; b]$ has a finite length.

Therefore, the class $\mathbb{BV}$ (and also $\mathbb{CBV}$) is a kind of "pseudo" algebraic field. And, for any function $f : [0; 1] \to [0; 1]$, it doesn't make any essential difference if we consider the length of its graph instead of its variation.

In some sense, the continuity and the property of bounded variation of a function are independent. Namely, there are both functions which are continuous but not of bounded variation and the functions which are of bounded variation but not continuous. However, for those functions which are both continuous and of bounded variation, we can show some significant results. For example, if both f and its differential f' are continuous, then $f \in \mathbb{BV}$ and actually, $V(f)[0;x] = \int_0^x |f'(u)|du$. This can be translated immediately to that of computable functions.

Lemma 3.2 *If both f and its derivative f' are computable on $[0;1]$, then $f \in \mathbb{CBV}$ and the function v_f defined by $v_f(x) := V(f)[0;x]$ is computable.*

Proof. $f \in \mathbb{CBV}$ follows directly from corresponding classical result and the computability of v_f follows from the fact that $v_f(x) = \int_0^x |f'(u)|du$ and the computability of the integral operation. $\square$

Another observation is that the variation of a continuous function can be calculated from the dyadic subdivisions instead of all real number subdivisions. In general, for any function $f : [0;1] \to [0;1]$ and $A \subseteq \mathrm{dom}(f)$, we define the *variation $V(f,A)$ of f on A* as $\sup_A \sum_{i<m} |f(x_i) - f(x_{i+1})|$, where the supremum is taken over all subdivisions $0 \leq x_0 < x_1 < x_2 < \cdots x_m \leq 1$ with $x_i \in A$. f is said to be of *bounded variation* on A (denoted by $f \in BV(A)$) if $V(f,A)$ is finite.

Proposition 3.3 *If $f \in \mathbb{CBV}$, then $V(f)[0;1] = V(f,\mathbb{D})$.*

Proof. Suppose that $f \in \mathbb{CBV}$. Then f is uniformly continuous and of bounded variation on the interval $[0;1]$. There is a modulus function $\alpha : \mathbb{N} \to \mathbb{N}$ such that $|f(x) - f(y)| \leq 2^{-n}$ if $|x-y| \leq 2^{-\alpha(n)}$. Let $u := V(f)[0;1]$ and $\varepsilon >$ be any positive real number. The inequality $V(f)[0;1] \geq \sup_{\mathbb{D}} \sum_{i<m} |f(r_i) - f(r_{i+1})|$ holds trivially by the definition of variation. It suffices now to show that $V(f)[0;1] \leq \sup_{\mathbb{D}} \sum_{i<m} |f(r_i) - f(r_{i+1})|$ holds too.

By definition of $V(f)[0;1]$, there is a subdivision $0 = x_0 < x_1 < x_2 < \cdots < x_m = 1$ such that $\sum_{i<m} |f(x_i) - f(x_{i+1})| > u - \varepsilon/2$. Fix an $n \in \mathbb{N}$ such that $m \cdot 2^{-n+1} < \varepsilon/2$ and choose, for all $i \leq m$, some $r_i \in \mathbb{D}$ such that $|x_i - r_i| \leq 2^{-\alpha(n)}$. Then, $\sum_{i<m} |f(r_i) - f(r_{i+1})| \geq \sum_{i<m}(|f(x_i) - f(x_{i+1})| - |f(r_i) - f(x_i)| - |f(x_{i+1}) - f(r_{i+1})|) \geq u - \varepsilon/2 - 2m \cdot 2^{-n} \geq u - \varepsilon$. This implies that $\sup_{\mathbb{D}} \sum_{i<m} |f(r_i) - f(r_{i+1})| \geq u = V(f)[0;1]$. $\square$

By Proposition 3.3, we can calculate the variation $V(f)[0;1]$ by simply considering all subdivisions of $[0;1]$ which consists of dyadic numbers instead of all real number subdivisions. This is especially useful if the function f is computable. In this case, we can approximate the variation $V(f)[0;1]$ from below effectively. Therefore $V(f)[0;1]$ is left computable. Unfortunately, it is not computable in general.

Lemma 3.4 *1. If $f \in \mathbb{CBV}$, the $V(f)[0;1]$ is a left computable real number;*

2. For any $y \in \mathbf{LC}$, there is a $f \in \mathbb{CBV}$ such that $V(f)[0;1] = c$.

Proof. Item 1. follows easily from Proposition 3.3.

To prove item 2., let $y \in \mathbf{LC}$ and $(y_s)_{s\in\mathbb{N}}$ an increasing computable sequence of rational numbers which converges to y. Define a function $f : [0;1] \to [0;1]$ as an infinite polygon function such that, for any $n \in \mathbb{N}$, f takes the value 0 at the endpoints and takes the value $(y_{n+1} - y_n)/2$ at the middle point of the interval $[n/(n+1); (n+1)/(n+2)]$. Moreover, we define $f(1) := 0$. Obviously, f is a computable function such that $V(f)[0;1] = \sum_{n\in\mathbb{N}} V(f)[n/(n+1); (n+1)/(n+2)] = \sum_{n\in\mathbb{N}}(y_{n+1} - y_n) = \lim_{s\to\infty} y_s = y$. □

Next is a well known results in analysis.

Proposition 3.5 *For any $f \in \mathbb{BV}$ there are nondecreasing function g, h such that $f(x) = g(x) - h(x)$ for all $x \in [0;1]$. Moreover, if f is continuous, then g, h can also be continuous.*

Proof. Define $g, h : [0;1] \to [0;1]$ by

$$g(x) = (f(0) + v_f(x) + f(x))/2 \text{ and } h(x) = (f(0) + v_f(x) - f(x))/2.$$

Then $f = g - h$. Since $v_f(y) - v_f(x) = V(f)[x;y] \geq |f(y) - f(x)|$ for any $x \leq y$, both g and h are nondecreasing. □

Notice that the functions of g, h defined in the above proof are not computable in general even if f is computable because the function v_f can be non-computable by Lemma 3.4.2. Therefore, the result cannot be extended immediately to that of computable functions. However the claim is still true if we require in addition that $V(f)[0;1]$ is computable as shown in the next theorem which belongs essentially to Douglas Bridges [2].

Theorem 3.6 *Let $f \in \mathbb{CBV}$. Then the following hold.*

1. $V(f)[0;1]$ is a computable real number if and only if v_f is a computable function; and

2. $V(f)[0;1]$ is computable, then there are two computable nondecreasing function $g, h : [0;1] \to [0;1]$ such that $f(x) = g(x) - h(x)$ for any $x \in [0;1]$.

Next result shows that the classes $\mathbb{BV}$ and $\mathbb{CBV}$ are not closed under the composition.

Lemma 3.7 *There are $f, g \in \mathbb{CBV}$ such that their composition $f \circ g \notin \mathbb{CBV}$.*

Proof. We define f by $f(x) := \sqrt{x}$ and define $g : [0;1] \to [0;1]$ as an infinite polygon function such that, for any $n \in \mathbb{N}$, g takes the value 0 at the endpoints and takes the value n^{-2} at the middle point of the interval $[n/(n+1);(n+1)/(n+2)]$. Furthermore, let $f(1) := 0$. Obviously, both f and g are computable functions. Their variation on $[0;1]$ are $V(f)[0;1] = f(1)-f(0) = 1$ and $V(g)[0;1] = \sum_{n\in\mathbb{N}} V(g)[n/(n+1);(n+1)/(n+2)] = \sum_{n\in\mathbb{N}} n^{-2} = 2$, respectively. Therefore $f, g \in \mathbb{CBV}$. On the other hand, their composition $f \circ g$ has an infinite variation $V(f\circ g)[0;1] = \sum_{n\in\mathbb{N}} V(\circ g)[n/(n+1);(n+1)/(n+2)] = \sum_{n\in\mathbb{N}} n^{-1} = \infty$. $\square$

4 $\mathbb{CBV}$(**LC**) and **WC**

By Theorem 2.3, the image of a left computable real number under CBV-function is weakly computable. In this section we will show that not every weakly computable real number is such an image. Namely, we will construct a computable sequence (x_s) of rational numbers converging weakly effectively to x which is not an image of left computable real number under CBV-functions.

To ensure that $x \notin \mathbb{CBV}(\mathbf{LC})$, we have to diagonalize all CBV-function and all left computable function. However, the CBV-functions are very difficult to deal with directly in an effective construction, we will use instead their approximation with effective error-estimation. The determinator introduced in next definition is one of such approximation.

Definition 4.1 Let $f : [0;1] \to [0;1]$, $m : \mathbb{N} \to \mathbb{N}$, $\beta : \mathbb{Q} \times \mathbb{N} \to \mathbb{Q}$ and $A \subseteq [0;1]_{\mathbb{Q}}$. β is a *determinator* of f if

$$\forall n \in \mathbb{N} \forall u \in \mathbb{Q}(|\beta(u,n) - f(u)| \leq 5^{-(n+1)}). \tag{2}$$

and β is simply a *determinator* if

$$\forall n \in \mathbb{N} \forall u \in \mathbb{Q}(|\beta(u,n) - \beta(u,n+1)| \leq 5^{-(n+1)}). \tag{3}$$

m is a *modulus* of f if $\forall x, y \in \mathbb{R}(|x-y| \leq 2^{-m(n)} \Longrightarrow |f(x)-f(y)| \leq 5^{-(n+1)})$. m is an A-*modulus* of β or a *modulus of* β *on* A if

$$\forall u, v \in A(|u-v| \leq 2^{-m(n)} \Longrightarrow |\beta(u,n) - \beta(v,n)| \leq 5^{-(n+1)}). \tag{4}$$

W.l.o.g. we assume in this papers that any (A-)modulus is increasing.

For the determinator β, we can define its variation as follows.

Definition 4.2 For any computable $\beta : [0;1] \times \mathbb{N} \to [0;1]$ and $A \subseteq [0;1]_{\mathbb{Q}}$, the *variation* $V(\beta, A)$ of β on A is defined by $\sup_A \sum_{i<k} |\beta(x_i, i) - \beta(x_{i+1}, i+1)|$, where the supremum is taken over all subdivision $0 \leq x_0 < x_1 \cdots < x_k \leq 1$ with $x_i \in A$. β is said of *bounded variation on* A ($\beta \in \mathbb{CBV}[A]$ for short) if $V(\beta, A)$ is finite. β is said of *bounded variation* if it is of bounded variation on $[0;1]_{\mathbb{Q}}$ and is denoted by $\beta \in \mathbb{CBV}$.

The following lemma follows immediately from the definition.

Lemma 4.3 *Let $\beta : [0;1]_{\mathbb{Q}} \times \mathbb{N} \to [0;1]_{\mathbb{Q}}$ be a determinator of a computable function $f : [0;1] \to [0;1]$ and $m, m' : \mathbb{N} \to \mathbb{N}$ be functions such that $m'(n) := m(n) + 2$. Then*

(1) $\lim_{s\to\infty} \beta(x_s, s) = f(\lim_{s\to\infty} x_s)$ *for any convergent sequence $(x_s)_{s\in\mathbb{N}}$.*
(2) f is of bounded variation iff β is of bounded variation on $\mathbb{Q}$ and iff β is of bounded variation on $\mathbb{D}$;
(3) If m is a modulus of f, then m' is a modulus of β on $\mathbb{D}$ and.
(4) If m is a modulus of β on $\mathbb{D}$, then m' is a modulus of f.

By effective Weierstrass Theorem (cf. [7]), it follows that any computable real function $f : [0;1] \to [0;1]$ has a computable determinator and a computable modulus. The inverse holds in fact also as shown in next Lemma.

Lemma 4.4 *If β is a computable determinator and α is a computable modulus of β, then there is a computable real function f such that β is a determinator of f and α is a modulus of f;*

Proof. We construct a type-2 Turing machine M to computable f as follows. For any input $(r_s)_{s\in\mathbb{N}}$ of rational sequence, M outputs a sequence $(t_s)_{s\in\mathbb{N}}$ of rational numbers, where $t_s := \beta(r_{2\alpha(s+1)}, s+1)$ for any $s \in \mathbb{N}$.

If $(r_s)_{s\in\mathbb{N}}$ is a fast convergent Cauchy sequence, i.e., $\forall s\,(|r_s - r_{s+1}| \leq 2^{-(s+1)})$, which converges to some $x \in \mathbb{R}$. Then $|r_{2\alpha(s+1)} - r_{2\alpha(s+2)}| \leq 2^{-2\alpha(s+1)} < 2^{-\alpha(s+1)}$. Because α is a modulus of β, this implies that $|\beta(r_{2\alpha(s+1)}, s+1) - \beta(r_{2\alpha(s+2)}, s+1)| < 5^{-(s+1)}$. Therefore, $(t_s)_{s\in\mathbb{N}}$ converges also effectively because

$$\begin{aligned} |t_s - t_{s+1}| &= |\beta(r_{2\alpha(s+1)}, s+1) - \beta(r_{2\alpha(s+2)}, s+2)| \\ &\leq |\beta(r_{2\alpha(s+1)}, s+1) - \beta(r_{2\alpha(s+2)}, s+1)| + \\ &\quad\ |\beta(r_{2\alpha(s+2)}, s+1) - \beta(r_{2\alpha(s+2)}, s+2)| \\ &\leq 5^{-(s+2)} + 5^{-(s+2)} \leq 2^{-(s+1)} \end{aligned}$$

Obviously, if $(r'_s)_{s\in\mathbb{N}}$ is another Cauchy sequence which converges effectively to

x, then the corresponding $(t'_s)_{s\in\mathbb{N}}$ converges effectively to the limit $\lim_{s\to\infty} t_s$ too. Therefore, M do compute a total real function f. It is also not difficult to see that β is a determinator of f and α is a modulus of f. $\square$

Now we are going to investigate how often oscillates a CBV-function around a small interval. More precisely, for any continuous function $f : [0; 1] \to [0; 1]$ and $I := [a; b] \subset [0; 1]$, we call a pair (x_1, x_2) of real numbers of $[0; 1]$ a *crossing* of f over I if $f(x_1)$ and $f(x_2)$ locate on the different sides of the interval I, i.e., $\max\{f(x_1), f(x_2)\} > b$ and $\min\{f(x_1), f(x_2)\} < a$. Denoted by $z(f, I, A)$, for $A \subseteq [0; 1]$, the number of the crossings of f over I on A, namely

$$\begin{aligned} z(f, I, A) := \max\{n \in \mathbb{N} : (\exists (x_i)_{i\le n} \in A^{n+1}(0 \le x_0 < x_1 \cdots < x_n \le 1) \ \& \\ \forall i < n((x_i, x_{i+1}) \text{ is a crossing of } f \text{ over } I)\}. \end{aligned}$$

For $A = [0; 1]$, we denote $z(f, I, A)$ simply by $z(f, I)$. Furthermore, for $\beta : [0; 1]_{\mathbb{Q}} \times \mathbb{N} \to [0; 1]_{\mathbb{Q}}$, we define accordingly $\hat{z}(\beta, I, n)$, for $n \in \mathbb{N}$, as the maximal natural number m such that there are rational numbers $0 \le x_0 < x_1 < x_2 < \cdots < x_m \le 1$ and that $\beta(x_s, n + s)$ and $\beta(x_{s+1}, n + s + 1)$ locate on different sides of the interval I for any $s < m$.

Let $f \in \mathbb{CBV}$ and $I \subseteq [0; 1]$. Divide I equally into n subintervals $[I]^n_i$ $(i < n)$ of length $l(I)/n$. Then we have $\sum_{i<n} z(f, [I]^n_i)l(I)/n \le V(f)[0; 1]$ for any $n \in \mathbb{N}$. Therefore, if n is large enough, then there must be some $i < n$ such that $z(f, [I]^n_i)l(I)/n$ very small. Namely, we have the following lemma.

Lemma 4.5 *If $f \in \mathbb{CBV}$ and $I \subseteq [0; 1]$, then*

$$(\forall e)\,(\exists N)\,(\forall n \ge N)\,(\exists i < n)\,(z(f, [I]^n_i)l(I)/n \le 2^{-e}) \tag{5}$$

For $\beta \in \mathbb{CBV}$ we can show a similar estimation about $\hat{z}(\beta, I, n)$. Moreover, this estimation is even effective if we known a modulus function α of β in advance.

Lemma 4.6 *Let $\beta \in \mathbb{CBV}$ be a determinator, α a computable modulus of β and $I \subset [0; 1]$. Then, for any n with $5^{-n} \le l(I)$,*

$$\hat{z}(\beta, I, n) \le z(\beta(\cdot, n), [I]^5_2, \mathbb{D}_{\alpha(n)}) \ \& \ z(\beta(\cdot,\, n), I, \mathbb{Q}) \le \hat{z}(\beta, [I]^5_2, n). \tag{6}$$

Furthermore, for any e, there exist $m_0 > 0$ and t_0 with $0 < t_0 < 5^{m_0} - 1$ such that $z(\beta(\cdot,\, n + m_0), [I]^{5^{m_0}}_{t_0}, \mathbb{Q}) \cdot l(I) \cdot 5^{-m_0} < 2^{-e}$.

Proof. Because $\beta \in \mathbb{BV}$, both $i := \hat{z}(\beta, I, n)$ and $j := z(\beta(\cdot, n), [I]^5_2, \mathbb{D}_{\alpha(n)})$ exist. By the definition of $\hat{z}$, there are rational numbers $0 \le x_0 < x_1 <$

$x_2 < \cdots < x_i \leq 1$ such that $\beta(x_t, n+t)$ and $\beta(x_{t+1}, n+t+1)$ locate on different sides of I for any $t < i$. Let $I = [a_0; a_5]$ and $I_s := [I]^5_s = [a_s; a_{s+1}]$ for $s < 5$ and each I_s is divided equally further by $(a_{st})_{t\leq 5}$. Suppose w.l.o.g. that $\beta(x_{2s}, n+2s) \leq a_0$ and $\beta(x_{2s+1}, m+2s+1) \geq a_5$.

Because β is a determinator, $|\beta(x_{2s}, n+2s) - \beta(x_{2s}, n)| \leq \sum_{t<2s} 5^{-(n+t+1)} \leq 2 \cdot 5^{-(n+1)}$. This implies that $\beta(x_{2s}, n) \leq a_0 - 2 \cdot 5^{-(n+1)} \leq a_{02}$ and similarly $\beta(x_{2s+1}, n) \geq a_{43}$. For any $x_s \in \mathbb{Q}$, there is an $x'_s \in \mathbb{D}_{\alpha(n)}$ such that $|x_s - x'_s| \leq 2^{-\alpha(n)}$. Since α is a modulus of β, this implies that $|\beta(x_s, n) - \beta(x'_s, n)| \leq 5^{-(n+1)}$ hence that $\beta(x'_{2s}, n) \leq a_{12} < a_2$ and $\beta(x'_{2s+1}, n) \geq a_{33} > a_3$. It follows that the pairs (x'_s, x'_{s+1}) are crossing of $\beta(\cdot, n)$ over $[I]^5_2$. By definition, this means that $z(\beta(\cdot, n), [I]^5_2, \mathbb{D}_{\alpha(n)}) \geq i = \hat{z}(\beta, I, n)$. $\square$

Notice that, $z(\beta(\cdot, n), [I]^5_2, \mathbb{D}_{\alpha(n)})$ can be computed effectively. Thus, we can bound $\hat{z}(\beta, I, n)$ effectively.

Theorem 4.7 $\mathbb{CVB}(\mathbf{LC}) \subsetneq \mathbf{WC}$.

Proof. The inclusion part is quite straightforward. Here we prove only the inequality part. Namely we show that there is a weakly computable real number y such that $y \neq f(x)$ for any left computable real number $x \in [0; 1]$ and any computable real function $f : [0; 1] \to [0; 1]$ of bounded variation.

Let $(\alpha_s)_{s\in\mathbb{N}}$, $(\beta_s)_{s\in\mathbb{N}}$ and $(\gamma_s)_{s\in\mathbb{N}}$ be effective enumerations of all computable functions $\alpha_s :\subseteq \mathbb{N} \to \mathbb{N}$, $\beta_s :\subseteq \mathbb{Q} \times \mathbb{N} \to \mathbb{Q}$ and $\gamma_s :\subseteq \mathbb{N} \to \mathbb{Q}$, respectively. We will construct effectively a computable sequence $(y_s)_{s\in\mathbb{N}}$ of rational numbers converging weakly effectively to some y which satisfies, for any $i, j, k \in \mathbb{N}$, the following requirement $R'_{\langle i,j,k\rangle}$:

> If β_i is a determinator of some $f \in \mathbb{CBV}$; α_j is a modulus of f and $(\gamma_k(s))_{s\in\mathbb{N}}$ converges to x_k increasingly, then $y \neq f(x_k)$.

By Lemma 4.3, the requirements $R'_{\langle i,j,k\rangle}$ can be replaced by the following requirements $R_{\langle i,j,k\rangle}$.

$$R_{\langle i,j,k\rangle}: \qquad \left.\begin{array}{l} \beta_i \text{ is a total determinator,} \\ \alpha_j \text{ and } \gamma_k \text{ are increasing} \\ V(\beta_i, \mathbb{Q}) \leq \langle i,j,k\rangle, \text{ and} \\ \alpha_j \text{ is a } \mathbb{D}\text{-modulus of } \beta_i \end{array}\right\} \Longrightarrow \lim_{s\to\infty} \beta_i(\gamma_k(s), s) \neq y.$$

The strategy to satisfy a single requirement R_e ($e = \langle i,j,k\rangle$) is quite simple. We fix a rational interval $I \subseteq [0; 1]$ as a base interval and divide I into subintervals $I_i := [I]^5_i$ (for $i < 5$). Then either I_1 or I_3 is a witness interval of R_e in the sense that each of its element satisfies the requirement R_e. Actually, we can find such witness interval effectively in finite steps as follows. Let's take I_1 as the first (default) candidate of the witness. If there are no s and t such that $\beta_{i,s}(\gamma_{k,s}(t), t) \in I_1$, then I_1 is a correct witness interval of R_e. Otherwise, if $\beta_{i,s_0}(\gamma_{k,s_0}(t_0), t_0) \in I_1$ for some s_0 and t_0, then take I_3 as a new candidate of witness interval. If there are $s_1 > s_0$ and $t_1 > t_0$ such that $\beta_{i,s_1}(\gamma_{j,s_1}(t_1), t_1) \in I_3$, let I_1 be the candidate of witness interval again, and so on. If $V(\beta_i, \mathbb{Q}) \leq e$, these can happen at most finitely often and we can find finally a correct witness interval of R_e. This procedure is called an I-jump game.

To satisfy all requirements simultaneously, we implement different I-jump games simultaneously and construct a sequence $(I_e)_{e\in\mathbb{N}}$ of nested rational intervals such that $I_e \subset I_{e-1}$, and I_{e-1} and I_e are base and witness intervals of R_e, respectively. Then the element $y \in \bigcap_{e\in\mathbb{N}} I_e$ satisfies all the requirements R_e. Unfortunately, such a sequence $(I_e)_{e\in\mathbb{N}}$ of witness intervals is not computable, because it is not effectively decidable, whether I_e is a correct witness interval of R_e. However, we can construct its effective approximation. Namely, we construct effectively the finite sequences $(I_{e,s})_{e\leq u(s)}$ such that $I_{e,s}$ is a correct witness interval of R_e for $(\beta_{i,s}, \alpha_{j,s}, \gamma_{k,s})$ instead of $(\beta_i, \alpha_j, \gamma_k)$, $I_{e+1,s} \subset I_{e,s}$ for any $e < u(s)$ and $\lim_{s\to\infty} u(s) = \infty$. In addition, we demand that the limit $I_e := \lim_{s\to\infty} I_{e,s}$ exists and it is a correct witness interval of R_e and $\lim_{e\to\infty} l(I_e) = 0$. Thus, $\bigcap_{e\in\mathbb{N}} I_e$ contains an unique element y which satisfies all requirements R_e. Let y_s be the middle point of $I_{u(s),s}$. Then, $(y_s)_{s\in\mathbb{N}}$ is a computable sequence of rational numbers which converges to y.

For any $e \in \mathbb{N}$, the intervals $I_{e,s}$ can be defined in stages by above strategy. To this connection, the priority injury technique can be applied. We say that a requirement R_e has a higher priority then $R_{e'}$ if $e < e'$. At any stage s, if several requirements require to define their current witness intervals, then we choose only one requirement R_e of the highest priority and define a new witness interval $I_{e,s}$ (R_e receives attention). Now, all $I_{e',s}$ for $e' > e$, if it is defined, do not work any more and have to be redefined at a later stage ($R_{e'}$ is injured at stage s). Since the I-jump game changes $I_{e,s}$ for R_e only finitely many times, every requirement R_e can be injured finitely often. This ensures that the correct witness interval I_e of R_e can be eventually found, namely, $I_e := \lim_{s\to\infty} I_{e,s}$ exists. To guarantee that I_e is nested, we choose $I_{e,s}$ always as a subinterval of $I_{e-1,s}$.

To guarantee that the sum $\sum_{s\in\mathbb{N}} |y_s - y_{s+1}|$ is finite, a more sophisticated strategy for satisfying R_e is necessary. Concretely, let S_e denote the set of all e-stages s at which the requirement R_e receives attention. Then, we demand that $\sum_{s+1\in S_e} |y_s - y_{s+1}| \leq 2^{-e}$ for any $e \in \mathbb{N}$. In other words, the e-stages contribute to the sum $\sum_{s+1\in\mathbb{N}} |y_s - y_{s+1}|$ at most 2^{-e}. To this end, we divide the base interval I_{e-1} into 5^m (for some $m > 1$) instead of 5 subintervals I_t $t < 5^m$

and find three intervals I_{t-1}, I_t and I_{t+1} such that $\hat{z}(\beta_i, I_t, n) \cdot l(I_t) \leq 2^{-e}$ for appropriate n. Then choose I_{t-1} or I_{t+1} as candidate of witness interval for R_e. This ensures that the I_{e-1}-jump game contributes to the sum $\sum_{s\in\mathbb{N}} |y_s - y_{s+1}|$ at most 2^{-e}. By Lemma 4.5 and Lemma 4.6, such intervals I_{t-1}, I_t and I_{t+1} can be effectively determined. Considering the possible injuring, we choose the intervals I_{t-1}, I_t and I_{t+1} in such a way that $\hat{z}(\beta_i, I_t, n) \cdot l(I_t) \leq 2^{-(b_e+e)}$, where b_e is the number of injuries the requirement R_e received up to now.

Before the formal construction, we explain our notations again. For any $e = \langle i, j, k\rangle \in \mathbb{N}$, the interval $I_{e-1,s}$ is the base interval of R_e at the stage s and it has the length $5^{-c_{e-1,s}}$. To apply the $I_{e-1,s}$-jump game, we choose three abutting subintervals $I^1_{e,s}, I^2_{e,s}, I^3_{e,s} \subset I_{e-1,s}$ of the same length $5^{-c_e(s)}$ and set one of the intervals $I^1_{e,s}$ and $I^3_{e,s}$ as a witness interval of R_e which is denoted by $I_{e,s}$. We call R_e is in the state of active if $I^1_{e,s}, I^2_{e,s}, I^3_{e,s}$ are defined. Denote by u_s the maximal index e such that $I_{e,s}$ is defined. Let $d_e := c_e - c_{e-1}$. Then $l(I_e) = l(I_{e-1}) \cdot 5^{-d_e}$. The number of injuries that R_e received up to stage s is denoted by $b_{e,s}$. At any step of an I_{e-1}-jump game, a new element $\gamma_{k,s}(t)$ of the sequence $(\gamma_k(t))_{t\in\mathbb{N}}$ is considered. Denoted by $a_{e,s}$ the maximal index of element of $(\gamma_k(s))_{s\in\mathbb{N}}$ which is already considered up to stage s. The triple $(\alpha_{i,s}, \beta_{j,s}, \gamma_{k,s})$ is called *n-proper* if it satisfies the premise of $R_{\langle i,j,k\rangle}$ on the set $\mathbb{D}_{\alpha_{i,s}(n)}$, namely, for any $u, v \in \mathbb{D}_{\alpha_i(n)}$, the following hold

$$
\begin{aligned}
&\forall m \leq n(|u-v| \leq 2^{-\alpha_i(m)} \Longrightarrow |\beta_{j,s}(u,m) - \beta_{j,s}(v,m)| < 5^{-(m+1)})\\
&\forall m < n\,(\alpha_{i,s}(m) \downarrow = \alpha_{i,s}(m+1))\ \&\ \forall m < n\,(\gamma_{k,s}(m) < \gamma_{k,s}(m+1))\\
&\forall m < n(|\beta_{j,s}(u,m) - \beta_{j,s}(u,m+1)| < 5^{-(m+1)}) \text{ and}\\
&V(\beta_{j,s}, \mathbb{D}_{\alpha_j(n)}) \leq \langle i,j,k\rangle\,.
\end{aligned}
$$

We initialize a requirement R_e by setting it into the state inactive, define $b_{e,s+} := b_{e,s} + 1$ and set all functions a_e, c_e, d_e and intervals I_e, I^v_e (for $v \in \{0,1,2\}$) undefined.

The formal construction of $(y_s)_{s\in\mathbb{N}}$:

Stage $s = 0$: Define $I_{-1} := (0;1)$, $u_0 := -1$ and all requirements R_e are initialized.

Stage $s+1$: Given u_s and $a_{e,s}, b_{e,s}, c_{e,s}, d_{e,s}$, $I_{e,s}$, $I^1_{e,s}, I^2_{e,s}, I^3_{e,s}$ for $e \leq u_s$ and $y_s = \mathrm{mid}(I_{u_s})$. A requirement R_e ($e = \langle i,j,k\rangle$) *requires attention of type I* (or I-attention) if R_e is in the state of inactive, $e \leq u_s$ and there are n_0, n_1 and t_0 which satisfies the following conditions

(I.1) $c_{e-1,s} < n_1 < n_0 < s$ and $0 < t_0 < 5^{m_0} - 1$ for $m_0 := n_1 - c_{e-1,s}$;
(I.2) $(\alpha_{i,s}, \beta_{j,s}, \gamma_{k,s})$ is n_0-proper; and
(I.3) $z(\beta_j(\cdot, n_1), J, \mathbb{D}_{\alpha_i(n_0)}) \cdot 5^{-n_1+1} \leq 2^{-(e+b_{e,s}+1)}$, where $J := [[I_{e-1,s}]^{5^{m_0}}_{t_0}]^5_2$.

A requirement R_e ($e = \langle i, j, k\rangle$) *requires attention of type II* (or II-attention) if R_e is in the state of active and there is a natural number $a > a_{e,s}$ such that $(\alpha_{i,s}, \beta_{j,s}, \gamma_{k,s})$ is a-proper and $\beta_{j,s}(\gamma_{k,s}(a), a) \in I_{e,s}$.

If no requirement requires attention (neither type I nor type II), then choose an $n > 1$ such that $l(I_{u_s,s})/5^n < 2^{-(u_s+1+b_{u_s+1,s}+1)}$ and define $u_{s+1} := u_s+1$. For $e := u_{s+1}$, define also $a_{e,s+1} := -1$, $b_{e,s+1} := 0$, $c_{e,s+1} := c_{u_s,s} + n$, $d_{e,s+1} := n$, $I_{e,s+1} := [I_{u_s,s}]^{5^n}_t$ for $t := 2 \cdot 5^{n-1} + 2$ and $y_{s+1} := \mathrm{mid}(I_{e,s+1})$. All other parameters remain unchanged. This stage is called a (default) u_{s+1}-stage and $R_{u_{s+1}}$ receives (default) attention at this stage.

Otherwise choose a minimal e_0 such that R_{e_0} requires attention. We consider the following two cases.

Case 1. R_{e_0} requires I-attention. Let m_0 and t_0 be the natural numbers which satisfy condition (I.3). Then we define $u_{s+1} := e_0$, $I^v_{e_0,s+1} := [I_{e_0-1,s}]^{5^{m_0}}_{t_0-1+v}$ for $v = 0, 1, 2$, $I_{e_0,s+1} := I^0_{e_0,s+1}$ and

$$a_{e_0,s+1} := n_1, \quad c_{e_0,s+1} := n_1, \quad d_{e,s+1} := m_0 \tag{7}$$

where $n_1 := c_{e-1,s} + m_0$. Set the requirement R_{e_0} into the state of *active* and initialize all R_e for $e > e_0$.

Case 2. R_{e_0} requires II-attention. Let a be the natural number such that $\beta_{j,s}(\gamma_{k,s}(a), a) \in I_{e,s}$. Define $u_{s+1} := e_0$, $a_{e_0,s+1} := a$, and $I_{e_0,s+1} := I^2_{e_0,s}$ if $I_{e_0,s} = I^0_{e_0,s}$ and $I_{e_0,s+1} := I^0_{e_0,s}$ otherwise. All requirements R_e for $e > e_0$ is initialized.

In both cases define $y_{s+1} := \mathrm{mid}(I_{e_0,s+1})$. The requirement R_{e_0} receives attention and all requirements R_e for $e > e_0$ are *injured* at this stage. This stage is called an e_0-stage.

To show our construction successes, we prove the following sublemmas.

Sublemma 4.7.1 *The requirement R_e requires and hence receives attentions finitely often for any $e \in \mathbb{N}$.*

Proof. We prove the sublemma by induction on $e \in \mathbb{N}$. Suppose by induction hypothesis that $R_{e'}$ requires and receives attentions finitely often for any $e' < e$. Choose a minimal s_0 such that no $R_{e'}$ for $e' < e$ requires attention after stage s_0. By the minimality of s_0, we have either $s_0 = 0$ or there is an $e' < e$ such that $R_{e'}$ receives attention at stage s_0. This means that R_e is in the state inactive at stage s_0.

If R_e requires no attention after stage s_0, then we are done. Otherwise, suppose that R_e requires and hence receives its first attention at stage $s_1 > s_0$. Namely we define the new c_{e,s_1} and new intervals I_{e,s_1} and I^v_{e,s_1} for $v \in \{0,1,2\}$ of the same length $5^{-c_{e,s_1}}$. R_e is set into the state active and will remain in this state forever. After stage s_1, R_e can require and receive only II-attentions at, say, $t_0+1 < t_1+1 < t_2+1 < \cdots$. For any s, we have $\beta_{j,t_s}(\gamma_{k,t_s}(a_{e,t_s+1}), a_{e,t_s+1}) \in I_{e,t_s}$ and $I_{e,t_s+1} = I^{|2-v|}_{e,t_s}$ if $I_{e,t_s} = I^v_{e,t_s}$ for some $v \in \{0,1\}$. Because the distance between the intervals I^0_{e,t_s} and I^2_{e,t_s} is $5^{-c_{e,s_1}}$, this implies that

$$|\beta_{i,t_s}(\gamma_{k,t_s}(a_{e,t_s+1}), a_{e,t_s+1}) - \beta_{i,t_{s+1}}(\gamma_{k,t_{s+1}}(a_{e,t_{s+1}+1}), a_{e,t_{s+1}+1})| \geq 5^{-c_{e,s_1}}$$

for any $s \in \mathbb{N}$. Let $s' := e \cdot 5^{c_{e,s_1}} + 1$. Then

$$\begin{aligned} &\sum_{t \leq t_{s'}} |\beta_{i,t_{s'}}(\gamma_{k,t_{s'}}(t), t) - \beta_{i,t_{s'}}(\gamma_{k,t_{s'}}(t+1), t+1)| \\ \geq &\sum_{s \leq s'} |\beta_{i,t_{s'}}(\gamma_{k,t_{s'}}(a_{e,t_s+1}), a_{e,t_s+1}) - \beta_{i,t_{s'}}(\gamma_{k,t_{s'}}(a_{e,t_{s+1}+1}), a_{e,t_{s+1}+1})| > e \end{aligned}$$

Therefore, R_e does not require attention any more after stage $t_{s'}$. Thus, R_e receives attentions finitely often altogether. □ (sublemma)

Sublemma 4.7.2 *For any $e \in \mathbb{N}$, limits $I_e := \lim_{s\to\infty} I_{e,s}$, $a_e := \lim_{s\to\infty} a_{e,s}$, $b_e := \lim_{s\to\infty} b_{e,s}$, $c_e := \lim_{s\to\infty} c_{e,s}$ and $d_e := \lim_{s\to\infty} d_{e,s}$ exist.*

Proof. By Sublemma 4.7.1, we can choose the minimal s_0 such that no $R_{e'}$ for $e' \leq e$ receives attention after stage s_0. According to the construction, either $s_0 = 0$ or there is an $e' \leq e$ such that $R_{e'}$ receives attention at stage s_0. If s_0 is an e-stage at which the interval I_{e,s_0} is defined, then it is in fact the last e-stage and hence $I_{e,s} = I_{e,s_0}$ for any $s \geq s_0$ since no $R_{e'}$ for $e' \leq e$ receives attention after stage s_0. Otherwise, if s_0 is an e'-stage for some $e' < e$, then there is a default e-stage $s > s_0$ at which the interval $I_{e,s}$ is defined and this interval will never be changed again. Therefore, in both cases, the limit $I_e := \lim_{s\to\infty} I_{e,s}$ exists.

Similarly, we can show that all other limits exist. □ (sublemma)

Sublemma 4.7.3 *For any $e \in \mathbb{N}$, the interval I_e witnesses the requirement R_e in the sense that any $y \in I_e$ satisfies R_e.*

Proof. Suppose that the premise of the requirement R_e (for $e = \langle i,j,k\rangle$) holds. By Sublemma 4.7.2, we can choose a minimal s_0 such that $I_{e,s_0} = I_{e,s} = I_e$ and $a_{e,s_0} = a_{e,s} = a_e$ hold for any $s \geq s_0$. Then the interval I_{e,s_0} is defined and will never be changed again after stage s_0. Especially, $u_s > e$ holds and the requirements R_e requires no attention at any stage $s \geq s_0$.

If R_e is in the state inactive at stage s_0, then it remains in state inactive at any stage $s \geq s_0$. This means that R_e requires no I-attention after stage s_0.

By assumptions on $(\alpha_i, \beta_j, \gamma_k)$ we have $\beta_i \in \mathbb{CBV}$. Therefore, by of Lemma 4.6, there are $n_1 > c_{e-1}$ and t_0 with $0 < t_0 < 5^{m_0} - 1$ for $m_0 := n_1 - c_{e-1}$ such that $z(\beta_j(\cdot, n_1), [I]_{t_0}^{5^{m_0}}, \mathbb{Q}) \leq 2^{-(e+b_e+1)}$. This implies that

$$z(\beta_j(\cdot, n_1), [[I]_{t_0}^{5^{m_0}}]_2^5, \mathbb{D}_{\alpha_i(n_0)}) \leq 2^{-(e+b_e+1)}$$

for any n_0. Choose $s_1 > n_0 > n_1$ large enough such that $(\alpha_{i,s_1}, \beta_{j,s_1}, \gamma_{k.s_1})$ is n_0-proper. Then R_e requires require attention at stage $s_1 + 1$. This contradicts the choice of s_0.

Otherwise, if R_e is in the state of active at stage s_0. Assume by contradiction that $\lim_{s\to\infty} \beta_i(\gamma_k(s)) = y := \lim_{s\to\infty} y_s$. By the definition of (y_s), we have obviously that $y \in I_e$. There is an s_1 such that $\beta_i(\gamma_k(s)) \in I_e$ for any $s \geq s_1$. Let $a := \max\{a_e + 1, s_1\}$. Since $(\alpha_j, \beta_i, \gamma_k)$ satisfies the premise of R_e, there is an $s_2 > a$ such that $(\alpha_{i,s_2}, \beta_{j,s_2}, \gamma_{k,s_2})$ is a-proper. Then R_e requires attention of type II at stage $s_2 + 1$ which is again a contradiction. This contradiction implies that $\lim_{s\to\infty} \beta_i(\gamma_k(s), s) \notin I_e$. Thus R_e is satisfied by any element of I_e. $\square$ (sublemma)

Sublemma 4.7.4 $\sum_{s\in\mathbb{N}} |y_s - y_{s+1}| \leq 4$, *hence* $y := \lim_{s\to\infty} y_s$ *is a weakly computable real number which is not in* $\mathbb{CVB}(\mathbf{LC})$.

Proof. Let S_e be the set of all e-stages. By the construction, every stage s is an e-stage for some $e \in \mathbb{N}$, hence $\bigcup_{e\in\mathbb{N}} S_e = \mathbb{N}$ and $\sum_{s\in\mathbb{N}} |y_s - y_{s+1}| = \sum_{e\in\mathbb{N}} \sum_{s+1\in S_e} |y_s - y_{s+1}|$. Therefore it suffices to prove that $\sum_{s+1\in S_e} |y_s - y_{s+1}| \leq 2^{-e+1}$ for any $e \in \mathbb{N}$.

By Sublemma 4.7.1, it is easy to see that S_e is finite for any e. There is an $u_0 \in \mathbb{N}$ such that $s_0 < s_1 < s_2 < \cdots < s_{u_0}$ are all elements of $\bigcup_{e'<e} S_{e'}$. In other words, for any $t \leq u_0$, s_t is an e'-stage for some $e' < e$ and hence R_e is injured at this stage. It is not difficult to see that $b_{e,s} = t$ for $s_t \leq s < s_{t+1}$. Let $S_e^t := \{s \in S_e : s_t < s < s_{t+1}\}$ for any $t < u_0$. Then, it suffices to show that

$$\forall t < u_0 \,(\sum_{s+1\in S_e^t} |y_s - y_{s+1}| \leq 2^{-(e+t)+1}). \tag{8}$$

Suppose that $t_0 + 1 < t_1 + 1 < \cdots < t_{v_0} + 1$ are all elements of S_e^t. Since R_e is injured at stage s_t, R_e is in the state of inactive at stage s_t and hence also at stage t_0. Therefore, stage $t_0 + 1$ must be a default e-stage because it is the first e-stage after R_e is injured at stage s_t. At this stage, we define a new default witness interval $I_{e,t_0+1} := [I_{e-1,t_0}]_t^{5^n}$ for some n and $t := 2 \cdot 5^{n-1} + 2$. We define y_{t_0+1} as the middle point of I_{e,t_0+1} which is obviously also the middle point of I_{e-1,t_0+1}. Therefore, $y_{e,t_0} = y_{e,t_0+1}$. Remember that we have choose n in such a way that the length $l(I_{e,t_0+1})$ of the new witness interval less or equal to

$2^{-(e+t+1)}$.

At stage $t_1 + 1$, R_e receives I-attention since I_{e,t_1} is defined (it cannot be a default e stage any more) and R_e is in the state of inactive (it requires no II-attention). Namely, we put R_e into the state active and define three consecutive intervals I^0_{e,t_1+1}, I^1_{e,t_1+1} I^2_{e,t_1+1}, and the interval $I_{e,t_1+1} = I^0_{e,t_1+1}$ such that they have the same length 5^{-n_1} and satisfy the following condition

$$z(\beta_i(\cdot, n_1), [I^1_{e,t_1+1}]^5_2, \mathbb{D}_{\alpha(n)}) \cdot 5^{-n_1} \leq 2^{-(e+t+1)} \tag{9}$$

for some $n > n_1$ is some natural number. Now y_{t_1+1} is the middle point of I_{e,t_1+1}. Notice that there is no e'-stage between stages $t_0 + 1$ and $t_1 + 1$. The witness intervals at stage $t_0 + 1$ and t_1 are the same, i.e., $I_{e,t_1} = I_{e,t_0+1}$. Therefore $|y_{t_1} - y_{t_1+1}| \leq 2^{-(e+t+1)}$, because $y_{t_1}, y_{t_1+1} \in I_{e,t_1} = I_{e,t_0+1}$. In addition, we define also $a_{e,t_1+1} := n_1$ so that only the elements of the sequence $(\gamma_k(s))_{s\in\mathbb{N}}$ which has the indices bigger than n_1 will be considered at the later stages.

At stage $t_v + 1$ for $1 < v \leq v_0$, R_e receives II-attention. By a simple induction we can show that $I_{e,t_{2v+1}+1} = I^0_{e,t_1+1}$, $I_{e,t_{2v+2}} = I^2_{e,t_1+1}$. Therefore, we have $|\beta_{j,t_v}(\gamma_{k,t_v}(a_{e,t_v+1})) - \beta_{j,t_{v+1}}(\gamma_{k,t_{v+1}}(a_{e,t_{v+1}+1}))| \geq 5^{-n_1}$ for any $v \geq 2$, because $\beta_{j,t_v}(\gamma_{k,t_v}(a_{e,t_v+1})) \in I_{e,t_v}$. Of course, this can happens at most $\hat{z}(\beta_i, I^1_{e,t_1+1}, n_1)$ times. Moreover, by the condition (9) and (6) of Lemma 4.6, it follows that

$$\begin{aligned}
&\sum_{1<v\leq v_0} |y_{t_v} - y_{t_c+1}| \leq \hat{z}(\beta_i, I^1_{e,t_1+1}, n_1) \cdot 5^{-n_1} \cdot 2 \\
&\leq z(\beta_i(\cdot, n_1), [I^1_{e,t_1+1}]^5_2, \mathbb{D}_{\alpha(n)}) \cdot 5^{-n_1} \cdot 2 \leq 2^{-(e+t)}.
\end{aligned}$$

Combining the case of $t_1 + 1$, (8) follows immediately. □ (sublemma)

By Sublemma 4.7.4, the real number y is a weakly computable real number which is not in $\mathbb{CVB}(\mathbf{LC})$. This completes the proof of the theorem □

5 $\mathbb{CBV}^2(\mathbf{LC})$ and $\mathbb{CTF}(\mathbf{LC})$

In the last section we show that one application of the CBV-functions to semi-computable real numbers produces a proper subset of weakly computable real numbers. In this section we will show that two applications of CBV-functions suffice to produce the set $\mathbb{CTF}(\mathbf{LC})$.

Theorem 5.1 $\mathbb{CVB}^2(\mathbf{LC}) = \mathbb{CTF}(\mathbf{LC}) = \mathbb{CTF}(\mathbf{WC})$

Proof. The inclusion $\mathbb{CVB}^2(\mathbf{LC}) \subseteq \mathbb{CTF}(\mathbf{LC})$ is trivial because $\mathbb{CBV}^2 \subseteq \mathbb{CTF}$. We prove now that $\mathbb{CTF}(\mathbf{LC}) \subseteq \mathbb{CVB}^2(\mathbf{LC})$ holds too.

Given any $y \in \mathbb{CTF}(\mathbf{LC})$, there is a computable sequence $(y_s)_{s\in\mathbb{N}}$ of rational numbers converging to y such that, for any $n \in \mathbb{N}$, its n-divergence is bounded by $b(n)$, for some recursive function $b : \mathbb{N} \to \mathbb{N}^*$. Now, we are going to construct two computable functions $g, h \in \mathbb{CBV}$ and an increasing computable sequence $(x_s)_{s\in\mathbb{N}}$ of rational numbers such that $g \circ h(x) = y$. Thus, $y \in \mathbb{CBV}^2(\mathbf{LC})$.

By definition, if $f \in \mathbb{CBV}$, then the sum $\sum |f(u_s) - f(u_{s+1})|$ is finite, for any partition $x_0 < x_1 < \cdots < x_k$ of $[0;1]$. That is, f can only make few big jumps or make many small jumps on the interval $[0;1]$. Since $\mathbb{CVB}(\mathbf{LC}) \neq \mathbb{CTF}(\mathbf{LC})$, the composition $g \circ h$ cannot be of bounded variation. That is, to satisfy $gh(x) = y$, the function $g \circ h$ has to make a lot of big jumps but not g and h. The essential idea here is that, let h make a lot of small jumps and then let g amplify them to the big jumps.

We define the functions by Lemma 2.6 again. Let $\delta_1(n) := 2b(3n)+1$, $\delta_2(n) := 2^{n+1} \cdot \prod_{i\leq n}(b(3i)+1)+1$ and $\delta_3(n) := 2$ be computable functions. I_1 and I_2 are canonical δ_1- and δ_2-interval trees, respectively. We define I_3 as a δ_3-interval tree in such a way that, for any $w \in \{0,1\}^*$ and $i \in \{0,1\}$, the interval $I_3(w)$ is covered by intervals $I_3(w0)$ and $I_3(w1)$ which are overlapped in the middle of the interval $I_3(w)$ for a length of $l(I(w))/4$. More precisely, the interval $I_3(w) := [a^{\delta_3}_w; b^{\delta_3}_w]$ has the length of $l^{\delta_3}_w := \prod_{i<|w|}(2^{-1} + 2^{-3i})$ for $a^{\delta_3}_w, b^{\delta_3}_w$ defined inductively by

$$\begin{cases} a^{\delta_3}_\lambda := 0, & b^{\delta_3}_\lambda := 1 \\ a^{\delta_3}_{w0} := a^{\delta_3}_w, & a^{\delta_3}_{w1} := a^{\delta_3}_w + l^{\delta_3}_w \cdot 3/8 \\ b^{\delta_3}_{w1} := b^{\delta_3}_w & b^{\delta_3}_{w0} := a^{\delta_3}_w + l^{\delta_3}_w \cdot 5/8 \end{cases}$$

Moreover, we define two computable functions $\iota_1 : \mathbb{N}^*_{\delta_1} \to \mathbb{N}^*_{\delta_2}$ and $\iota_2 : \mathbb{N}^*_{\delta_2} \to \mathbb{N}^*_{\delta_3}$ inductively by $\iota_1(\lambda) := \lambda$, $\iota_2(\lambda) := \lambda$ and

$$\iota_1(wi) := (\iota_1(w)1 \text{ if } i \bmod 4 = 1;\ \iota_1(w)3 \text{ if } i \bmod 4 = 3;\ \uparrow \text{ otherwise})$$
$$\iota_2(wi) := (\iota_2(w)0 \text{ if } i = 1;\ \iota_2(w)1 \text{ if } i = 3;\ \uparrow \text{ otherwise, })$$

for any $w \in \mathbb{N}^*$ and. Obviously, ι_1 and ι_2 are also (δ_1, δ_2)- and (δ_2, δ_3)-compatible, respectively. By Lemma 2.6, there are computable real functions $g, h : [0;1] \to [0;1]$ such that

$$h(I_1(w)) \subseteq I_2(\iota_1(w)) \ \&\ g(I_2(u)) \subseteq I_3(\iota_2(u)) \tag{10}$$

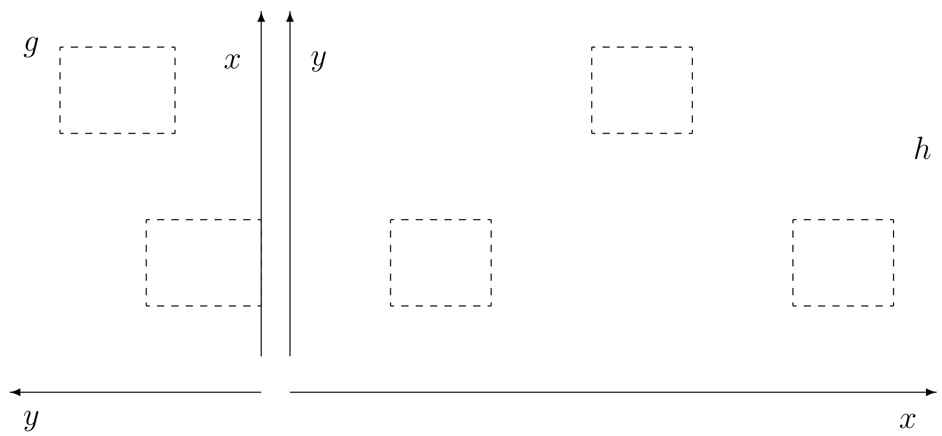

Fig. 1. An example of g and h to their second approximation step

for any $w \in \mathrm{dom}(\iota_1)$ and $u \in \mathrm{dom}(\iota_2)$.

The Figure 1 gives an example of g and h to their second approximation step, namely, the condition (10) is hold for $|w| = |u| = 2$. Moreover, g, h have both of finite variations because

$$V(g)[0;1] = \sum_{n\in\mathbb{N}} 2^n \cdot 2^{-3n+1} \leq 2 \quad \text{and}$$
$$V(h)[0;1] \leq \sum_{n=0}^{\infty} \prod_{i\leq n} 2\delta_2^{-1}(b(3i)+1) \leq \sum_{n=0}^{\infty} 2^{-n} = 2.$$

It remains to construct an increasing computable sequence $(x_s)_{s\in\mathbb{N}}$ of rational numbers converging to $x \in \mathbf{LC}$ such that $y = \lim_{s\to\infty} gh(y_s) = gh(x)$. Notice that $\iota_1(w) \in \mathrm{dom}(\iota_2)$ and hence $gh(a_w^{\delta_1}) = g(a_{\iota_1(w)}^{\delta_2}) = a_{\iota_1\iota_2(w)}^{\delta_3}$ for any $w \in \mathrm{dom}(\iota_1)$. In the following, we will construct a computable sequence $(w_s)_{s\in\mathbb{N}}$ of $\mathbb{N}_{\delta_1}^*$ and let $x_s := a_{w_s}^{\delta_1}$.

The formal construction of the sequences $(w_s)_{s\in\mathbb{N}}$.

Stage $s = 0$. Define simply $w_0 = \lambda$ and hence $x_0 := 0$. The stage 0 is called a default stage.

Stage $s + 1$. Given $w_s \in \mathrm{dom}(\iota_1)$. Suppose that $y_s \in I_3(\iota_2\iota_1(w_s))$. Then define $w_{s+1} := w_s1$ if $y_s \in I_3(\iota_2\iota_1(w_s1))$ and $w_{s+1} := w_s3$ otherwise. In this case, the stage $s + 1$ is also a default stage.

Suppose that $y_s \notin I_3(\iota_1\iota_2(w_s))$. Then choose a maximal $n \leq |w_s|$ such that $y_s \in I_3(\iota_1\iota_2(w_s) \upharpoonright n)$ and let $w_{s+1} = (w_s \upharpoonright n)(w_s[n] + 2)$. Notice that, $\iota_1(w_{s+1})[n] = 1$ if $\iota_1(w_s)[n] = 3$ and $\iota_1(w_{s+1})[n] = 3$ otherwise. This implies that $\iota_2\iota_1(w_{s+1})[n] = 1 \dot{-} \iota_2\iota_1(w_s)[n]$, and hence $y_s \in I_3(\iota_2\iota_1(w_{s+1})[n])$. In this case, we call the stage $s + 1$ an n-stage.

We will show that the construction successes. Notice that at any stage $s+1$, we have defined w_{s+1} either by $w_{s+1} := w_s i$ for $i \in \{1, 3\}$ or $w_{s+1} := (w_s \upharpoonright n)(w_s[n]+2)$ for some $n < |w_s|$. In both cases we have $a^{\delta_1}_{w_s} < a^{\delta_1}_{w_{s+1}}$. Therefore, the sequence $(x_s)_{s\in\mathbb{N}}$ defined by $x_s := a^{\delta_1}_{w_s}$ is increasing. Of course, we have to prove that this sequence is well defined at all. Namely, we show the next claim.

Claim 1 *For any $n \in \mathbb{N}$, the following hold*

(1) There are at most finitely many n-stages in the construction; and
(2) $\forall s \in \mathbb{N}\, (n < |w_s| \Longrightarrow w_s[n] < \delta_1(n))$. Namely, $w_s \in \mathbb{N}^*_{\delta_1}$ for any s.

We prove two assertions by induction on n simultaneously. Assume by induction hypothesis that, for all $m < n$, there are at most finitely many m-stages and $w_s[m] < \delta_1(m)$ holds whenever $w_s[m]$ is defined. Let S_n be the set of all n-stages s. Then $S_{<n} := \bigcup_{m<n} S_m$ is a finite set by the assumption. Suppose that $s_1 < s_2 < \cdots < s_k$ are all elements of $S_{<n}$. Since $0 \notin S_{<n}$, we have $0 < s_1$. Denote $s_0 := 0$ and $s_{k+1} := \infty$. Then it suffices to show that $w_s \in \mathbb{N}^*_{\delta_1}$ for any $t < k+1$ and $s_t \leq s < s_{t+1}$.

Given $t < k+1$, there are no m-stage (for $m < n$) between stages s_t and s_{t+1}. Therefore, $w_s \upharpoonright n = w_{s_t} \upharpoonright n$ for any s between s_t and s_{t+1} and $w_s[n]$ can be changed only at some n-stages s for $s_t < s < s_{t+1}$. Let $t_0+1 < t_1+1 < t_2+1 < \cdots$ are all n-stages between stages s_t and s_{t+1}. Since s_t is an m-stage for some $m < n$, $w_{s_t}[n]$ is not defined. There is a default stage t_0 between stages s_t and $t_0 + 1$ at which $w_{t_0}[n] \in \{1, 3\}$ is defined. Suppose w.l.o.g. that $w_{t_0}[n] = 3$. At any n-stages $t_i + 1$ for $i \geq 1$, the value $w_{t_i+1}[n]$ will be increased by two. It suffices now to show that there are at most $b(3n)$ n-stages between stages s_t and s_{t+1}. By a simply induction on $j \in \mathbb{N}$, we can show that the following hold.

$$\iota_2\iota_1(w_{t_{2j}})[n] = 1 \;\;\&\;\; \iota_2\iota_1(w_{t_{2j+1}})[n] = 0 \tag{11}$$

$$y_{t_{2j}} \in I_3(\iota_2\iota_1((w_{t_{2j}} \upharpoonright n)1) \;\;\&\;\; y_{t_{2j+1}} \in I_3(\iota_2\iota_1((w_{t_{2j+1}} \upharpoonright n)0) \tag{12}$$

From (12) it follows that $|y_{t_i} - y_{t_{i+1}}| \geq 2^{-3n}$ because $w_{t_i} \upharpoonright n = w_{t_{i+1}} \upharpoonright n$. But this can happen at most $b(3n)$ times by the assumption on the sequence $(y_s)_{s\in\mathbb{N}}$. In other words, there are at most $b(3n)$ n-stages between s_t and s_{t-1}. This completes the proof of the claim.

By Claim 1, $(x_s)_{s\in\mathbb{N}}$ is a well defined increasing computable sequence of rational numbers which converges to some left computable real number $x \in [0; 1]$. On the other hand, it follows from (2) of the Claim 1 that the length of w_s convergence to infinite and accordingly the length of the intervals $I_3(\iota_2\iota_1(w_s))$ converges to 0. According to the construction, we know that both $gh(x_s)$ and y_s belong to the intervals $I_3(\iota_2\iota_1(w_s))$ for any $s \in \mathbb{N}$. This implies that

$gh(x) = \lim_{s\to\infty} gh(x_s) = \lim_{s\to\infty} y_s = y$ and hence that $y \in \mathbb{CVB}^2(\mathbf{LC})$. $\square$

Corollary 5.2 $\mathbb{CBV}(\mathbf{WC}) = \mathbb{CTF}(\mathbf{WC})$

6 $\mathbb{CMF}(\mathbf{WC})$ and $\mathbb{CTF}(\mathbf{WC})$

In this section we will show that the image sets of weakly computable real numbers under computable total real functions and under computable monotone real functions are different. Notice at first that $y \in \mathbb{CMF}(\mathbf{WC})$ iff there is a strictly monotone computable function f and a $x \in \mathbf{WC}$ such that $f(x) = y$. Furthermore, any strictly monotone computable function can be approximated by a computable sequence of strictly monotone rational functions.

Lemma 6.1 *Let $f : [0; 1] \to [0; 1]$ be a strictly increasing computable real function. Then there is a computable function $\beta : [0; 1]_{\mathbb{Q}} \times \mathbb{N} \to [0; 1]_{\mathbb{Q}}$ such that*

$$\forall n \forall x, y \in [0; 1]_{\mathbb{Q}} (x < y \Longrightarrow \beta(x, n) < \beta(y, n)) \tag{13}$$

$$\forall n \forall x \in [0; 1]_{\mathbb{Q}} (|\beta(x, n) - \beta(x, n + 1)| \leq 2^{-3(n+1)}) \tag{14}$$

The assertion holds for decreasing function accordingly

Lemma 6.2 *Let $f : [0; 1] \to [0; 1]$ be a strictly monotone function, $J \subseteq [0; 1]$ a non-empty rational interval. There is a $t_0 \in \mathbb{N}$ such that, for any $r \geq t_0$, there are rational numbers $a_1 < a_2 < a_3 < a_4$ belonging to J such that*

$$a_3 - a_2 \geq 2^{-(2r+2)} \text{ and } |f(a_1) - f(a_4)| \leq 2^{-(r+1)}. \tag{15}$$

Proof. For a strictly monotone function $f : [0; 1] \to [0; 1]$ and any nonempty rational interval $J := [b_1; b_2] \subseteq [0; 1]$, choose a $t_0 > 0$ such that

$$|f(b_2) - f(b_1)| / (b_2 - b_1) < 2^{t_0 - 1} \text{ and} \tag{16}$$

$$(b_2 - b_1)/2 < (4i_0 + 4)2^{-(2t_0+2)} < b_2 - b_1. \tag{17}$$

for some $i_0 \in \mathbb{N}$. Obviously, such t_0 exists. For any $s, i \in \mathbb{N}$, let

$$J_i^s := [b_1 + (4i + 1)2^{-(t_0+s+2)}; b_1 + (4i + 4)2^{-(t_0+s+2)}]. \tag{18}$$

Then we have $(b_2 - b_1)/2 < \max J_{i_0}^{t_0} < b_2$.

We will show that t_0 satisfies the lemma. Assume by contradiction that there is an $r \geq t_0$ such that no rational numbers $a_1 < a_2 < a_3 < a_4$ satisfy condition (15). For such an r, we define $a^i_j := b_1+(4i+j)\cdot 2^{(t_0+r+2)}$, for any $i \in \mathbb{N}$ and $j \in \{1,2,3,4\}$. Obviously, we have $a^i_1 < a^i_2 < a^i_3 < a^i_4$ and $|a^i_3 - a^i_2| = 2^{-(t_0+r+2)} \geq 2^{-(2r+2)}$. By the choice of r, this implies that $\max f(J^r_i) - \min f(J^r_i) = |f(a^i_1) - f(a^i_4)| > 2^{-(r+1)}$ as long as $J^r_i \subseteq J$. On the other hand, there are at least $(b_2 - b_1)\cdot 2^{t_0+r}(=(b_2-b_1)/4\cdot 2^{-(t_0+r+2)})$ such intervals J^r_i. We conclude that $|f(b_2) - f(b_1)|/(b_2-b_1) \geq (b_2-b_1)\cdot 2^{t_0+r}\cdot 2^{-(r+1)}/(b_2-b_1) \geq 2^{(t_0-1)}$. This contradicts the choice of t_0. □

Theorem 6.3 $\mathbb{CMF}(\mathbf{WC}) \subsetneq \mathbb{CBV}(\mathbf{WC}) = \mathbb{CTF}(\mathbf{WC})$

Proof. It suffices to prove the inequality part. We will construct a recursive function h and a computable sequence (y_s) of rational numbers converging to y and the n-divergence of (y_s) is bounded by $h(n)$ for any n. Thus $y \in \mathbf{DBC} = \mathbb{CTF}(\mathbf{WC})$. Furthermore, y satisfies, for all $i, j \in \mathbb{N}$, the requirement

$Q_{\langle i,j\rangle}$: If φ_i is a strictly monotone total function and γ_j is total such that $\sum_{s\in\mathbb{N}} |\gamma_j(s) - \gamma_j(s+1)| \leq 1$, then $\lim_{s\to\infty} \varphi_i(\gamma_j(s)) \neq y$,

where (φ_e) and (γ_e) are effective enumerations of all computable functions $\varphi_e :\subseteq [0;1] \to [0;1]$ and $\gamma_e :\subseteq \mathbb{N} \to [0;1]_{\mathbb{Q}}$. Thus, $y \notin \mathbb{CMF}(\mathbf{WC})$.

Given a strictly monotone function φ_i and a weakly convergent sequence $(\gamma_j(s))$ with $\sum_{s\in\mathbb{N}} |\gamma_j(s) - \gamma_j(s+1)| \leq 1$, we consider a base interval $J \subseteq [0;1]$. Let r and a_1, a_2, a_3, a_4 satisfy Lemma 6.2 and define $I^1 := [a_1; a_2]$, $I^2 := [a_3; a_4]$, $J^1 := \varphi_i(I^1)$ and $J^2 := \varphi_i(I^2)$. Now, if $\varphi_i(\gamma_j(s))$ enters J^1 (hence $\gamma_j(s)$ enters I^1), then we define y_{s+1} as the middle point of J^2. Similarly, if $\varphi_i(\gamma_j(s))$ enters J^2 (hence $\gamma_j(s)$ enters I^2), then we define y_{s+1} to be the middle point of J^1. This guarantees that the limits $y := \lim_{s\to\infty} y_s$ and $\lim_{s\to\infty} \varphi_i(\gamma_j(s))$ have at least a distance of $|\varphi_i(a_2) - \varphi_i(a_3)|$, hence y satisfies the requirement $Q_{\langle i,j\rangle}$. Notice that y_s's can be redefined according to this strategy at most 2^{2r+2} times because of $a_3 - a_2 \geq 2^{-(2r+2)}$ and the hypotheses $\sum_{s\in\mathbb{N}} |\gamma_j(s) - \gamma_j(s+1)| \leq 1$. On the other hand, every redefinition of y_s contributes only a jump of (y_s) bounded by $2^{-(r+1)}$ because of (15).

To satisfy all requirement Q_e simultaneously, let's begin with the base interval $I_0 := [0;1]$ and search for the minimal $e := \langle i,j\rangle$ such that we can apply Lemma 6.2 for the function φ_i. Choose r_1 and a_1, a_2, a_3, a_4 which satisfy Lemma 6.2 and let $I^1_e := [a_1; a_2]$, $I^2_e := [a_3; a_4]$ and $J^u_e := \varphi_i(I^u_e)$ for $u := 1, 2$. By default, let $I_1 := I^1_e$ be a new base interval, define y_{s_1} to be the middle point of J^1_e (denote by $\mathrm{mid}(J^1_e)$). If at a later stage $s_2 > s_1$, $\varphi_i(\gamma_j(s_2))$ enters the interval J^1_e, then set $I_1 := I^2_e$. If there is another $s_3 > s_2$ such that $\varphi_i(\gamma_j(s_2))$ enters the interval J^2_e, then redefine $I_1 := I^1_e$, and so on. In each case, we will define a new value of (y_s) as the middle point of I_1. Of course, this redefinition can appear at most $2^{(2r_1+1)}$ times if $\sum_{s\in\mathbb{N}} |\gamma_j(s) - \gamma_j(s+1)| \leq 1$.

Now on the base interval I_1 we will look for another minimal $e_1 := \langle i_1, j_1\rangle > e$ such that the Lemma 6.2 can be applied for φ_{i_1}. Define r_2, $I^u_{e_1}$, $J^u_{e_1}$ $(u := 1, 2)$, I_2 and new y_s similarly. This procedure can be carried out further. By the above strategy, we can see that, first, the limit $y := \lim_{s\to\infty} y_s$ exists. In fact it is the unique common point of a nested interval sequence $(I_e)_{e\in\mathbb{N}}$; Second, every requirement $Q_{\langle i,j\rangle}$ is satisfied, because $\lim_{s\to\infty} \varphi_i(\gamma_j(s))$ and y have at least the distance of $|\varphi_i(a_2) - \varphi_i(a_3)|$ (for some $a_2 < a_3$), if φ_i and γ_j satisfy the premise of $Q_{\langle i,j\rangle}$; Third, the n-divergence of (y_s) is bounded by a recursive function h defined by $h(n) := \sum_{m<n} 2^{2m+2}$. Where the third claim follows from the observation that we define new y_s only according to some requirement and some natural number r which satisfies Lemma 6.2 and any jump which is related to this r is not greater than $2^{-(r+1)}$. Different requirement relate to different such r and, for any fixed r, there are at most 2^{2r+2} jumps related to this r.

Unfortunately, the construction above is not effective, because, first, we cannot decide whether φ_i is a monotone total function and, second, we can't compute the value $\varphi_i(\gamma_j(s))$ in finite steps, even if it is defined. To solve this problem, let $\beta_i :\subseteq [0;1]_{\mathbb{Q}} \times \mathbb{N} \to [0;1]_{\mathbb{Q}}$ be an approximation of φ_i such that $|\varphi_i(x) - \beta_i(x,n)| \leq 2^{-n}$ and use the function pair $(\beta_{i,s}, \gamma_{j,s})$ in the construction in stead of (φ_i, γ_j). More precisely, let (β_e) and (γ_e) be effective enumerations of all computable functions $\beta_e :\subseteq [0;1]_{\mathbb{Q}} \times \mathbb{N} \to [0;1]_{\mathbb{Q}}$ and $\gamma_e :\subseteq \mathbb{N} \to [0;1]_{\mathbb{Q}}$, $(\beta_{e,s})$ and $(\gamma_{e,s})$ are their uniformly effective approximation, respectively. Then it suffices that the limit $y := \lim y_s$ satisfies, for all $i, j \in \mathbb{N}$ the following requirements.

$R_{\langle i,j\rangle}$: If β_i is a determinator of a strictly monotone total function, β_i satisfies conditions (13) and (14) and $\sum_{s\in\mathbb{N}} |\gamma_j(s) - \gamma_j(s+1)| \leq 1$, then $\lim_{s\to\infty} \beta_i(\gamma_j(s), s) \neq y$.

To satisfy the requirement $R_{\langle i,j\rangle}$, we have to guarantee that, for some positive $\epsilon > 0$, $|y_s - \beta_{i,s}(\gamma_{j,s}(m), m)| \geq \epsilon$ holds for infinitely many m, whenever the pair $(\beta_{i,s}, \gamma_{j,s})$ does not contradicts the premise of $R_{\langle i,j\rangle}$. We say that the pair (β_i, γ_j) is s-proper if $(\beta_{i,s}, \gamma_{j,s})$ does not contradicts the premise of $R_{\langle i,j\rangle}$, namely,

$$\forall n \forall u, v\, (u < v \Longrightarrow \beta_{i,s}(u,n) < \beta_{i,s}(v,n)) \tag{19}$$
$$\forall n \forall u\, (\beta_{i,s}(x,n) < \beta_{i,s}(x,n+1)) \tag{20}$$
$$\forall m \forall n \forall u\, (n < m \Longrightarrow \beta_{i,s}(x,m) - \beta_{i,s}(x,n) \leq 2^{-(n+1)}) \tag{21}$$
$$\sum_{t<k} |\gamma_{j,s}(s_t) - \gamma_{j,s}(s_{t+1})| \leq 1 \tag{22}$$

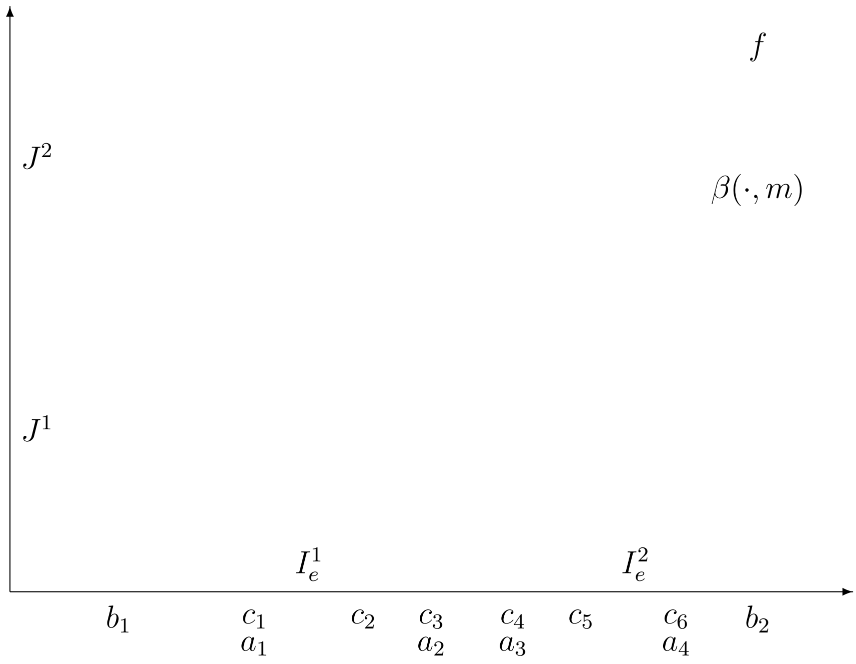

Fig. 2. The approximation $\beta_i(\cdot, m)$ of a function f

hold whenever the pertinent values of $\beta_{i,s}$ are defined. Where $(s_t)_{t\leq k}$ is an increasing enumeration of the domain $\mathrm{dom}(\gamma_{j,s})$.

Before we give the formal construction, let's explain some supplementary notations to be used in the construction. At any stage s, we will define a finite sequence of intervals: $I_{0,0} = [0; 1] \supset I_{1,s} \supset \cdots \supset I_{d_s,s}$. Each interval $I_{k,s}$ is defined according to some requirement $R_{e_s(k)}$ for any $k \leq d_s$. The definition of a new interval $I_{t,s}$ is related to some natural number $r_{t,s}$ which is determined by Lemma 6.2. For the effective estimation of n-divergence of (y_s), r's should be different for different intervals. Thus $r_{t,s} < r_{t+1,s}$ holds for any $t \leq d_s$. For some $t \leq d_s$, we define also two intervals $I^1_{e_s(t)}$ and $I^2_{e_s(t)}$ (and accordingly the intervals $J^1_{e_s(t)}$ and $J^1_{e_s(t)}$ of y-axis) for the requirement $R_{e_s(t)}$. Then we choose $I^1_{e_s(t)}$ or $I^2_{e_s(t)}$ alternatively as $I_{t,s}$ according to whether $\beta_i(\gamma_j(m), m)$ enters $J^2_{e_s(t)}$ or $J^1_{e_s(t)}$, where $e_s(t) = \langle i, j\rangle$. The value m used for R_e at stage s is denoted by $m_s(e)$. In the following, we often omit the index s if it is clear from the context. Besides, because we know only the approximation $\beta_{i,s}$ instead of the function φ_i, the definition of the intervals J^1_e and J^2_e is slightly different from that mentioned above. (see Fig 2.)

The formal construction.

Stage $s = 0$. We define $d_0 := 0, I_{0,0} := [0; 1]$, $e_0(0) := -1$, $r_{0,0} := 0$, $m_0(e) := 0$ and $y_0 := 1/2$. Stage 0 is called a 0-stage.

Stage $s + 1 = \langle i, j, k\rangle$, for some $i, j, k \in \mathbb{N}$. Let $e := \langle i, j\rangle$. If the pair (β_i, γ_j) is not s-proper, then go directly to the next stage. We suppose now that (β_i, γ_j)

is s-proper and consider the following cases.

Case 1. There is no $t_1 \leq d_s$ such that $e_s(t_1) = e$. But there exist a maximal $t \leq d_s$ and there are $r, m \in \mathbb{N}$ and $c_1, c_2, c_3, c_4, c_5, c_6 \in I_{e_s(t)}$ such that:

A1 $r_{t,s} < r < m < s$;
A2 $\min I_{e_s(t)} < c_1 < c_2 < c_3 < c_4 < c_5 < c_6 < \max I_{e_s(t)}$;
A3 $c_4 - c_3 \geq 2^{-(2r+2)}$;
A4 $\min\{|\beta_{i,s}(c_3, m) - \beta_{i,s}(c_2, m)|, |\beta_{i,s}(c_5, m) - \beta_{i,s}(c_4, m)|\} \geq 2^{-m}$,
A5 $|\beta_{i,s}(c_6, m) + 2^{-m} - \beta_{i,s}(c_1, m)| < 2^{-r}$.

Then we say that the requirement R_e *requires attention* (to be activated) at this stage. We define $d_{s+1} := t+1, e_{s+1}(t+1) := e, m_{s+1}(e) := m, r_{t+1,s+1} := r$. Set $e_{s+1}(t')$ and $r_{t',s+1}$ to be undefined for any $t' > t+1$. Furthermore, we define the intervals

$$\begin{cases} I^1_{e,s+1} := [c_1; c_2] \ \& \ I^2_{e,s+1} := [c_5; c_6] \\ J^1_{e,s+1} := [\beta_{i,s}(c_1, m);\ \beta_{i,s}(c_3, m)] \\ J^2_{e,s+1} := [\beta_{i,s}(c_5, m);\ \beta_{i,s}(c_6, m) + 2^{-m}]. \end{cases} \tag{23}$$

At last, let $I_{t+1,s+1} := I^1_e$ and $y_{s+1} := \mathrm{mid}(J^1_{e,s+1})$. The requirement R_e *receives attention* of activation and the stage $s+1$ is called an r-stage in this case. Notice that, the new intervals $J^1_{e,s+1}$ and $J^2_{e,s+1}$ defined in this case have the length $\leq 2^{-r}$ by (A5). Besides, y_s, y_{s+1} belong either to the interval $J^1_{e_s(t),s}$ or $J^2_{e_s(t),s}$, hence $|y_s - y_{s+1}| \leq 2^{-r_{t,s}}$.

Case 2. If $e = e_s(t)$ for some $t \leq d_s$ and there exists an $m > m_s(e)$ such that both y_s and $\beta_{i,s}(\gamma_{j,s}(m), m)$ belong to the same interval $J^1_{e,s}$ or $J^1_{e,s}$, then we say that R_e *requires attention* (to jump). In this case, we define $I_{t,s+1} := I^1_{e,s}$, $y_{s+1} := \mathrm{mid}(J^1_{e,s})$ if $y_s \in J^2_{e,s}$ and $I_{t,s+1} := I^2_{e,s}$, $y_{s+1} := \mathrm{mid}(J^2_{e,s})$ otherwise. Besides, set $d_{s+1} := t$. The interval $I_{t',s+1}$ and $e_{s+1}(t')$ are undefined now for any $t' > t$. The requirement R_e *receive attention* of jumping. The stage $s+1$ is called an $r_{t,s}$-stage. Notice that we have also $|y_s - y_{s+1}| \leq 2^{-r_{t,s}}$ in this case.

In both case 1 and 2, we call that the requirement $R_{e'}$ is *injured* at this stage if $e' = e_s(t')$ for some $t < t' \leq d_s$.

Case 3. If R_e does not require attention in any of above sense, then go directly to the next stage.

In all three cases, function values which is not redefined remain the same as in stage s. This completes the construction.

To show that our construction succeeds, we prove the following sublemmas.

Sublemma 6.3.1 *For any s and $t_1 < t_2 \leq d_s$, we have $e_s(t_1) < e_s(t_2)$.*

Proof. It follows immediately from the construction. □ (sublemma)

Sublemma 6.3.2 *For any e, the requirement R_e requires and receives attention finitely often.*

Proof. We prove the sublemma by induction on e. Suppose by induction hypothesis that all requirement $R_{e'}$ for $e' < e$ requires and receives attentions at most finitely often.

Choose a minimal s_0 such that no requirement $R_{e'}$ for $e' < e$ requires and receives attention after stage s_0. By the minimality of s_0, we have either $s_0 = 0$ or some $R_{e'}$ ($e' < e$) receives attention at stage s_0. Then it is easy to see that, for any $s \geq s_0$,

$$d_s \geq d_{s_0} \ \&\ I_{d_{s_0},s} = I_{d_{s_0},s_0} \ \&\ e_s(d_{s_0}) = e_{s_0}(d_{s_0}) < e. \tag{24}$$

If R_e requires no attention after stage s_0, then we are done. Otherwise, suppose that R_e requires and receives attentions after stage s_0 at stages $s_1+1 < s_2+1 < s_3+1 < \cdots$. We will show that this sequence is finite.

Obviously, R_e receives attention of activation at stage s_1+1. Namely, we choose the numbers $r_{d_{s_1+1},s_1+1}$, $m_{s_1+1}(e)$ and $c_1, \cdots, c_6$ which satisfy conditions (A1)–(A5) and define four intervals I^1_{e,s_1+1}, I^2_{e,s_1+1}, J^1_{e,s_1+1} and J^2_{e,s_1+1} according to (23). As default, we define also $I_{d_{s_1+1},s_1+1} := I^1_{e,s_1+1}$. Since R_e is never injured after stage s_0, we have $d_s \geq d_{s_1+1}$ and $e_s(d_{s_1+1}) = e$ for any $s \geq s_1+1$. Therefore, R_e can receive only attention of jumping and hence the intervals I^1_e, I^2_e, J^1_e and J^2_e will not be redefined after stage s_1+1. Notice also that, we define $y_{s_1+1} := \mathrm{mid}(I_{d_{s_1+1}}) = \mathrm{mid}(J^1_e)$.

For simplicity, let $m_t := m_{s_t+1}(e)$ for any $t \geq 1$ and $d := d_{s_1+1}$. At stage s_2+1, R_e requires attention to jump, that is, $\beta_i(\gamma_j(m_2), m_2) \in J^1_e$, because $I_{d_{s_2}} = I^1_e$ and hence $y_{s_2} \in J^1_e$. Therefore, we have $\beta_i(\gamma_j(m_2), m_1) < \beta_i(\gamma_j(m_2), m_2) \leq \beta_i(c_3, m_1)$. This implies $\gamma_j(m_2) \leq c_3$. At this stage, we redefine also $I_d := I^2_e$.

At stage s_3+1, R_e requires attention to jump again. This time we have $\beta_i(\gamma_j(m_3), m_3) \in J^2_e$ which implies $\beta_i(\gamma_j(m_3), m_3) \geq \beta_i(c_5, m_1) \geq \beta_i(c_4, m_1) + 2^{-m_1}$. Now we have $\gamma_j(m_3) \geq c_4$, otherwise, we can conclude a contradiction that $\beta_i(\gamma_j(m_3), m_3) \leq \beta_i(\gamma_j(m_3), m_1) + 2^{-m_1} < \beta_i(c_4, m_1) + 2^{-m_1}$.

In general, by an induction on $t \geq 1$, we can show that $\gamma_j(m_{2t}) \leq c_3$ & $c_4 \leq \gamma_j(m_{2t+1})$ whenever R_e receives attention at stage $s_{2t}+1$ and $s_{2t+1}+1$. On the other hand, because (β_j, γ_j) is s_t-proper for any such $t \geq 1$, we have

$\sum_{s<t} |\gamma_j(m_s) - \gamma_j(m_{s+1})| \leq 1$. Since $|c_4 - c_3| \geq 2^{-(2r_{s_1+1}+2)}$, this implies that R_e can receive attention at most $2^{2r_{s_1+1}+2}$ times after stage s_0. Therefore, R_e receives attention at most finitely often altogether. $\square$ (sublemma)

Sublemma 6.3.3 *For any $t \in \mathbb{N}$, $e(t) := \lim_{s\to\infty} e_s(t)$ exists. Thus, we have $\lim_{s\to\infty} d_s = \infty$ and the limit $y := \lim_{s\to\infty} y_s$ exists.*

Proof. We prove the sublemma by induction on t. Suppose by the induction hypothesis that $e(t') := \lim_{s\to\infty} e_s(t')$ exists for any $t' < t$. Choose an s_0 such that $e_s(t') = e_{s_0}(t')$ holds for all $t' < t$ and $s \geq s_0$. If there is an $s_1 \geq s_0$ such that $d_{s_1} \geq t$, then $e_{s_1}(t)$ is defined. Since $e_{s_1}(t)$ can be redefined only after the requirement $R_{e_{s_1}(t)}$ is injured by some requirement $R_{e'}$ of higher priority (i.e., $e' < e_{s_1}(t)$), $e_{s_1}(t)$ cannot be changed after stage s_1 because $e(t')$ ($t' < t$) is never changed after stage s_1 Therefore, $\lim_{s\to\infty} e_s(t) = e_{s_1}(t)$ exists.

Otherwise, suppose that $e_s(t)$ is never defined after stage s_0. Define a pair (β, γ) of functions by $\beta(x, n) := (1 - 2^{-n})x$ and $\gamma(n) := 2^{-n}$ for any $x \in [0;1]_{\mathbb{Q}}$ and $n \in \mathbb{N}$. Obviously, both β and γ are computable. By Padding Lemma, there are infinitely many pairs (i, j) such that $\beta = \beta_i$ and $\gamma = \gamma_j$. Choose a minimal $e := \langle i, j\rangle$ such that $e > e(t-1)$ and $\beta = \beta_j$ and $\gamma = \gamma_j$. It is not difficult to see that R_e will require and receive attention of activation at some stage $s > s_0$. Therefore, $e_s(t)$ will be defined at stage s. This is a contradiction. $\square$ (sublemma)

Sublemma 6.3.4 *For any $i, j \in \mathbb{N}$, if β_i is a determinator of a strictly monotone function $\varphi : [0;1] \to [0;1]$ and satisfies conditions (13) and (14), then there exists $t \in \mathbb{N}$ such that $e(t) = e$ for $e := \langle i, j\rangle$.*

Proof. Given β_i and j as in the Lemma. By Sublemma 6.3.3, there exists a maximal t_0 such that $e(t_0) < e$. Choose an s_0 such that no requirement $R_{e'}$ for $e' < e$ requires and receives attention after stage s_0 and $e_s(t) = e_{s_0}(t)$ holds for any $t \leq t_0$ and $s \geq s_0$. This implies also that $I_{t_0} := I_{t_0,s_0} = I_{t_0,s}$ and $r_{t_0} := r_{t_0,s_0} = r_{t_0,s}$ for any $s \geq s_0$.

By Lemma 6.2, for $J := I_{t_0}$, there are rational numbers $r > r_{t_0}$ and $a_1 < a_2 < a_3 < a_4$ from J which satisfy condition (15). Choose an $m > r$ such that $3 \cdot 2^{-m} < \min\{f(a_4) - f(a_3), f(a_3) - f(a_2), f(a_2) - f(a_1)\}$. Let $c_1 := a_1, c_3 := a_2, c_4 := a_3, c_6 := a_4$ and choose c_2, c_5 in such a way that $c_1 < c_2 < c_3$ & $c_4 < c_5 < c_6$, $\beta_i(c_3, m) - \beta_i(c_2, m) \geq 2^{-m}$ and $\beta_i(c_5, m) - \beta_i(c_4, m) \geq 2^{-m}$. (see Fig. 2.) By the choice of m, such c_2, c_5 exist.

Let $s_1 > s_0$ such that $\beta_{i,s_1}(\gamma_{j,s_1}(m), m)$ and $\beta_{i,s_1}(c_u, m)$ (for $u \in \{1, 2, \cdots, 6\}$) are all defined. Then, at some stage $s = \langle i, j, k\rangle \geq s_1$ (for some $k \in \mathbb{N}$) the requirement R_e requires and hence receives attention of activation. At this stage, we define $e_s(t_0 + 1) := e$. Since R_e is never injured after stage s, we have

$e(t_0+1) = e_s(t_0+1) = e$. $\square$ (sublemma)

Sublemma 6.3.5 *y satisfies all requirements R_e. Therefore, $y \notin \mathbb{CMF}(\mathbf{WC})$.*

Proof. Let $e = \langle i, j \rangle$. Suppose that β_i is a determinator of a strictly monotone function f, β_i satisfies conditions (13) and (14), $\sum_{s\in\mathbb{N}} |\gamma_j(s) - \gamma_j(s+1)| \leq 1$ and $z_e := \lim_{s\to\infty} \beta_i(\gamma_j(s), s)$ exists. We will show that $z_e \neq y$. First notice that the pair (β_i, γ_j) is s-proper for any s.

By Sublemma 6.3.4, there are t and s_0 such that $e = e(t) = e_s(t)$ for any $s \geq s_0$. Obviously, we have also $I_t = I_{t,s}$, $I_e^u = I_{e,s}^u$ and $J_e^u = J_{e,s}^u$ for $u = 1, 2$ and all $s \geq s_0$. Suppose that $I_t = I_e^1$ (the case for $I_t = I_e^2$ can be discussed similarly). Then by construction we have $y_s \in J_e^1$ for any $s \geq s_0$ and hence $y \in J_e^1$. In fact y must be an inner point of the interval J_e^1 because of condition A3.

Assume by contradiction that $y = z_e$. There is an $m > m_{s_0}(e)$ such that $\beta_i(\gamma_j(m), m) \in J_e^1$. Therefore, there exists an $s_1 > \max\{m, s_0\}$ such that $y_{s_1}, \beta_{i,s_1}(\gamma_{j,s_1}(m), m) \in J_{e,s_1}^1$. That is, the requirement R_e requires and hence receives attention at stage s_1+1. Thus $e_{s_1+1}(t)$ will be redefined at stage s_1+1. A contradiction. $\square$ (sublemma)

Sublemma 6.3.6 *Let $h(n) := \sum_{m<n} h_1(m)$, where h_1 is defined inductively by $h_1(0) := 1$ and, for any $n > 0$, $h_1(n) := \sum_{m<n} h_1(m) \cdot 2^{2n+2}$. For any $n \in \mathbb{N}$, the n-divergence of the sequence (y_s) is bounded by $h(n)$. Therefore y is divergence-bounded computable real numbers.*

Proof. By construction, a new element y_s is defined if and only s is an r-stage for some $r \in \mathbb{N}$. Furthermore, if y_{s+1} is defined at an r-stage $s+1$, then we have $|y_s - y_{s+1}| \leq 2^{-r}$. Denote by S_r the set of all r stages and $S_{<r} := \bigcup_{t<r} S_t$. Then the n-divergence of (y_s) is bounded by $|S_{<n}|$. Now we will show by an induction on n that $|S_n| \leq h_1(n)$ for any $n \in \mathbb{N}$. Suppose by induction hypothesis that $|S_m| \leq h_1(m)$ holds for any $m < n$. Let $s_0 < s_1 < \cdots < s_k$ be all elements of $S_{<n}$. From the proof of Sublemma 6.3.1, it is easy to see that, between any stages s_i and s_{i+1} (for $i < k$) or after stage s_k there are at most 2^{2n+2} n-stages. Thus, $|S_n| \leq |S_{<n}| \cdot 2^{2n+2} \leq \sum_{m<n} h_1(m) \cdot 2^{2n+2} = h_1(n)$. Therefore, we can conclude that $|S_{<n}| \leq \sum_{m<n} h_1(m) = h(n)$, that is, the n-divergence is bounded by $h(n)$. $\square$ (sublemma)

It is clear that the real number y is divergence bounded computable real numbers by Sublemma 6.3.6. Then $y \in \mathbb{CTF}(\mathbf{WC})$ by Theorem 2.3. On the other hand, by Sublemma 6.3.5, we know that $y \notin \mathbb{MON}(\mathbf{WC})$. This completes the proof of the theorem. $\square$

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