Effectiveness of the Global Modulus of Continuity on Metric Spaces

Klaus Weihrauch and Xizhong Zheng

Theoretische Informatik
FernUniversität Hagen
58084-Hagen, Germany
klaus.weihrauch@fernuni-hagen.de
xizhong.zheng@fernuni-hagen.de

Abstract. *Let* (X, d_X) *and* (Y, d_Y) *be metric spaces. By definition, there is a function* $h : (f, x, \epsilon) \mapsto \delta$, $(\delta > 0)$*, such that for all continuous function* $f : X \to Y$, $x \in X$ *and* $\epsilon > 0$: $\forall x' \in X(d_X(x, x') < \delta \Longrightarrow d_Y(f(x), f(x')) < \epsilon)$. *By a recent result of Repovš and Semenov [8], there is a function* h *continuous in* f, x *and* ϵ *with this property, if* (X, d_X) *is locally compact. Based on Weihrauch's frameworks on computable metric space ([13]), we effectivize this result by showing that there is a computable function of this type. The proof is a direct construction not depending on [8].*

Key words: Modulus of Continuity; Metric Space; Effective Analysis.

1 Introduction

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is continuous if, for any $x \in X$ and any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\forall x' \in X \ (d_X(x, x') < \delta \Longrightarrow d_Y(f(x), f(x')) < \epsilon). \tag{1}$$

In other words, f is continuous if and only if there is a (total) function $\tilde{\delta} : X \times \mathbb{R}^+ \to \mathbb{R}^+$ such that, for any $(x, \epsilon) \in X \times \mathbb{R}^+$,

$$\forall x' \in X \ (d_X(x, x') < \tilde{\delta}(x, \epsilon) \Longrightarrow d_Y(f(x), f(x')) < \epsilon). \tag{2}$$

The function $\tilde{\delta}$ is called a *modulus of continuity* of f.

The discussion about modulus of continuity is an interesting and important topic both in classical and effective analysis (see e.g, [2, 5, 10, 12]). For example, Ko [2] shown that, if $f : [a; b] \to \mathbb{R}^+$ is a computable (hence continuous) real function, then there is a recursive function $m : \omega \to \omega$ such that the function $\tilde{\delta}$ defined by $\tilde{\delta}(x, \epsilon) = 2^{-\lfloor 1/\epsilon \rfloor}$ is a modulus of (uniform) continuity of f. For the classically locally compact metric space X, Repovš and Semenov proved in [8] that, every continuous function $f : X \to Y$ possesses a continuous modulus of continuity. In fact, they have proved even more that, it is possible to determine an appropriate $\delta > 0$ satisfying (1) as a continuous function of the triple (f, x, ϵ) under proper topology. More precisely, let $C(X, Y)$ be the set of all continuous functions from X into Y, endowed with the topology of uniform convergence, i.e., the ϵ-neighbourhood of $f \in C(X, Y)$ is the set $B(f, \epsilon) := \{g \in C(X, Y) : \forall x \in X \ (d_Y(f(x), g(x)) < \epsilon\}$. Then the following has been proved in [8].

Theorem (Repovš and Semenov [8]). *Let (X, d_X) and (Y, d_Y) be metric spaces and suppose that X is locally compact. Then there exists a continuous function $\hat{\delta} : C(X, Y) \times X \times \mathbb{R}^+ \to \mathbb{R}^+$ such that, for every triple $(f, x, \epsilon) \in C(X, Y) \times X \times \mathbb{R}^+$,*

$$\forall x' \in X \ (d_X(x, x') < \hat{\delta}(f, x, \epsilon) \Longrightarrow d_Y(f(x), f(x')) < \epsilon). \tag{3}$$

The function $\hat{\delta}$ in above theorem is called *global modulus of continuity* of function space $C(X, Y)$. The proof of Theorem applies Michael's Selection Theorem (cf. [6]). That is, the function $\hat{\delta}$ exists as a selection function of some lower semi continuous set valued funtion. Thus it is quite ineffective.

Our main purpose of this paper is to discuss the effectiveness about the global modulus of continuity of $C(X, Y)$ when X is (effectively) locally compact. It is, of course, only possible after a computability framework about metric spaces is well established. In [13], Weihrauch introduced computability on metric space by the representation theory. In this theory, computability on finite and infinite sequences over some finite alphabet of symbols are defined explicitly, e.g., by Turing machines, and computability on other sets are introduced by representations, i.e., naming systems, where infinite sequences of symbols are used as names (cf, e.g. [3, 11, 12]). In this paper we will work in this framework and prove finally a computationally effective version of the above theorem, i.e. there is a computable global modulus of continuity of $C(X, Y)$. Thus, we can choose effectively a positive real δ satisfying (1) from any continuous function $f : X \to Y$, element $x \in X$ and positive real ϵ.

We fix Σ to be a finite alphabet containing all symbols we need. Σ^* and Σ^ω are the sets of all finite words and infinite sequences over Σ, respectively. The computability theory on Σ^* and Σ^ω have been well established ([9, 12]). To discuss the computability on other sets, we need their representations by the elements of Σ^* or Σ^ω. For any set A with the cardinality of at most continuum, a representation of A is simply a surjective function $\delta : \Sigma^a \to A$ for $a \in \{*, \omega\}$. For example, let $\Sigma = \{0, 1, \sharp, ¢\}$, the functions $\nu_N : \Sigma^* \to \omega$, $\nu_Q : \Sigma^* \to \mathbb{Q}$ and $\rho_R : \Sigma^\omega \to \mathbb{R}$ be defined respectively by

$$\nu_N(w) = n \iff w = 1^n;$$

$$\nu_{_Q}(w) = r \iff w = 0x0y0z0 \ \& \ r = \frac{\nu_{_N}(x) - \nu_{_N}(y)}{\nu_{_N}(z) + 1};$$

$$\rho_{_r}(p) = a \iff p = \sharp r_0 \sharp r_1 \sharp r_2 \sharp \ \ldots \ \ \& \lim_{n\to\infty} \nu_{_Q}(r_n) = a$$
$$\forall n \forall m \geq n(|\nu_{_Q}(r_n) - \nu_{_Q}(r_m)| < 2^{-n}).$$

Then $\nu_{_N}$, $\nu_{_Q}$ and $\rho_{_R}$ are the standard representations of the natural numbers ω, rational numbers$\mathbb{Q}$ and real numbers $\mathbb{R}$. By the representations, the fundemental concepts of classical computability theory can be translated directly. Suppose that δ_i are the representations of sets A_i for $i = 0, 1, \ldots, n$. An n-ary function $f : A_1 \times \cdots \times A_n \to A_0$ is called $(\delta_1, \ldots, \delta_n, \delta_0)$-computable if there is a computable function $g : (\Sigma^\omega)^n \to \Sigma^\omega$ such that $\forall p_1, \ldots, p_n(f(\delta_1(p_1), \ldots, \delta_n(p_n)) = \delta_0 \circ g(p_1, \ldots p_n))$.

Suppose that $\rho_{_R}$, $\rho_{_X}$ and $\delta_{_{XY}}$ be reasonable representations of $\mathbb{R}$, X and $C(X, Y)$, respectively. A computable metric space X is effectively locally compact, if we can determine effectively, for any $x \in X$, a neighbourhood B_x of x such that $\overline{B}_x$, the closure of B_x, is compact. Then our main result says that if X and Y are computable metric spaces (see definition of next section) and X is effectively locally compact, then there exists a $(\delta_{_{XY}}, \rho_{_X}, \rho_{_R}, \rho_{_R})$-computable function $\hat{\delta} : C(X, Y) \times X \times \mathbb{R}^+ \to \mathbb{R}^+$ which satisfies (3) for every triple $(f, x, \epsilon) \in C(X, Y) \times X \times \mathbb{R}^+$.

Our notations are almost same as in [15, 17]. For the notational simplicity, we often do not distinguish explicitly a natural number $i := \nu_{_N}(\bar{i})$ from its name $\bar{i} \in \Sigma^*$ under the standard representation $\nu_{_N}$. In this paper, $B(x, r)$ and $\overline{B}(x, r)$ denote always the open and closed r-ball with center x, respectively. As usual, $\langle \cdot, \cdot \rangle$ is the standard computable pairing function of natural numbers and π_1, π_2 are corresponding first and second reverse functions.

2 Computable Metric Spaces

In this section we review at first the notions of computable metric space and the corresponding representations of Weihrauch [13]. Then the representations of open and compact subsets of a computable metric space and the notion of effectively local compacteness will also be introduced.

Definition (Weihrauch[13]). Let (X, d) be a metric space, $A \subseteq X$ a countable dense subset of X and $\alpha : \omega \to A$ a (bijective) notation of A. If the set D defined by

$$D := \{\langle i, j, k, l \rangle \in \omega : \nu_{_Q}(k) < d(\alpha(i), \alpha(j)) < \nu_{_Q}(l)\}$$

is recursive enumerable, then we call (X, d, A, α), or simply X, a *computable metric space*. The open subset $U_k \subseteq X$ for $k \in \omega$ defined by:

$$U_{\langle i,j \rangle} := \{x \in X : d(\alpha(i), x) < \nu_{_Q}(j)\} \tag{4}$$

is called *basic open set* of X and $\nu_{_Q}(j)$ is its *radius* and denoted by $\mathrm{rd}(U_{\langle i,j \rangle})$ or simply $\mathrm{rd}(\langle i, j \rangle)$.

For example, suppose that d_R is the standard metric on $\mathbb{R}$ and $\alpha_Q : \omega \to \mathbb{Q}$ defined by $\alpha_Q(\langle i, j, k\rangle) = \frac{i-j}{k+1}$ is an enumeration of $\mathbb{Q}$. Then $(\mathbb{R}, d_R, \mathbb{Q}, \alpha_Q)$ is a computable metric space.

It is easy to see that the class of all basic open sets $\{U_i\}_{i\in\omega}$ is a topological basis of the topology deduced by the metric d. In our discussion, the basic open sets play an important role which is very similar to that of the rational intervals in effective analysis. Especially, the elements of the computable metric space and the continuous functions between the computable metric spaces can be represented by some infinite sequences of basic open sets.

Definition (Weihrauch[13]). Let (X, d_X, A_X, α_X) and (Y, d_Y, A_Y, α_Y) be the computable metric spaces. $En(p) := \{(i_0, i_1, \ldots\ i_m) : \sharp i_0 ¢ i_1 ¢ \ldots\ ¢ i_m \sharp \sqsubset p\}$ the set enumerated by sequence $p \in \Sigma^\omega$. We define the representations $\rho_X, \rho'_X : \Sigma^\omega \to X$ and $\delta_{XY}, \delta'_{XY} : \Sigma^\omega \to C(X, Y)$ respectively by

$$\rho_X(p) = x \iff p = \sharp i_0 \sharp i_1 \sharp i_2 \sharp \cdots \ \ \&\ \lim_{n\to\infty} \alpha_X(i_n) = x \ \& \\ \forall n \geq m\ (d_X(\alpha_X(i_n), \alpha_X(i_m)) < 2^{-m});$$

$$\rho'_X(p) = x \iff En(p) = \{i \in \omega : x \in U_i\};$$

$$\delta_{XY}(p) = f \iff p = \sharp i_0 ¢ j_0 \sharp i_1 ¢ j_1 \sharp i_2 ¢ j_2 \sharp \cdots \ \ \&\ \forall n\ (f(\overline{U}_{i_n}) \subseteq U_{j_n})\ \& \\ \forall x \in X\ \forall \epsilon > 0\ \exists \langle i, j\rangle\ (\sharp i ¢ j \sharp \sqsubset p\ \&\ x \in U_i\ \&\ \mathrm{rd}(j) < \epsilon);$$

$$\delta'_{XY}(p) = f \iff \forall q \in \mathrm{dom}(f \circ \rho_X)\ (f(\rho_X(q)) = \rho_Y(M_U(p, q))),$$

where M_U is the universal type-2 Turing machine.

As shown in [13], ρ_X is equvelent to ρ'_X and δ_{XY} is equivelt to δ'_{XY} with respect to the effective reduction. So we can use them exchangably. Because any open subset is a union of some basic open sets and any compact subset has a finite cover of basic open sets (we call it a basic finite open cover), we can define the representations of open and compact subsets through basic open sets similar to the case of $\mathbb{R}$ (cf [15, 17]).

Definition 1. Let (X, d, A, α) be a computable metric space. O_X and K_X be the sets of all open and compact subsets of X. Then we can define the representations $\theta_X : \Sigma^\omega \to O_X$ and $\kappa_X : \Sigma^\omega \to K_X$ by

$$\theta_X(p) = O \iff p = \sharp i_0 \sharp i_1 \sharp i_2 \sharp \cdots \ \ \&\ O = \bigcup\{U_{i_n} : n \in \omega\};$$

$$\kappa_X(p) = G \iff En(p) = \{i_0 ¢ \cdots\ ¢ i_n : n \in \omega\ \&\ G \subseteq \bigcup_{s\leq n} U_{i_s} \\ \forall s \leq n\ (U_{i_s} \bigcap G \neq \emptyset)\}.$$

Note that, $\theta_X(p) = O$ means that p enumerates a sequence of basic open sets which exhaust set O. If $\kappa_X(p) = G$ then p enumerates all finite basic open covers of G which intersects A. Next proposition shows that, for a κ-name p of G, it is equivalent to enumerate a Cauchy sequence of the finite basic open covers of G which intersrects G and has limit G in the sense of Hausdorff metric.

Proposition 2. *Let (X, d, A, α) be a computable metric space and d_H the Hausdorff metric defined by*

$$d_H(U, V) = \max\{\sup_{x\in U}\inf_{y\in V} d(x, y),\ \sup_{x\in V}\inf_{y\in U} d(x, y)\}$$

for any sets $U, V \subseteq X$, (cf. e.g.,[1]). If the representation κ'_X is defined by

$$\begin{aligned}\kappa'_X(p) = G \iff\ & p = \sharp i_0^0 \text{¢} \cdots \text{¢} i_{m_0}^0 \sharp i_0^1 \text{¢} \cdots \text{¢} i_{m_1}^1 \sharp \cdots \ \ \& \\ & \forall n\ (G \subseteq \bigcup\nolimits_{j\le m_n} U_{i_j^n}\ \&\ \forall s \le m_n (G \bigcap U_{i_s^n} \ne \emptyset))\ \& \\ & \forall t\ \forall s \ge t\ (d_H(\bigcup\nolimits_{j\le m_s} U_{i_j^s}\ ,\ \bigcup\nolimits_{j\le m_t} U_{i_j^t}) < 2^{-t})\end{aligned}$$

Then we have $\kappa_X \equiv \kappa_X$, i.e., there are computable functions $f, g : \Sigma^\omega \to \Sigma^\omega$ such that $\kappa = \kappa' \circ f$ and $\kappa' = \kappa \circ g$.

Proof. Let $\kappa_X(p) = G$ with $p = \sharp i_0^0 \text{¢} \cdots \text{¢} i_{m_0}^0 \sharp i_0^1 \text{¢} \cdots \text{¢} i_{m_1}^1 \sharp \cdots$, then p enumerates all finite basic open covers of G which intersects G. It is easy to see that we can construct a Turing machine M which enumerates, from the input p, a sequence $q := \sharp j_0^0 \text{¢} \cdots \text{¢} j_{m_0}^0 \sharp j_0^1 \text{¢} \cdots \text{¢} j_{m_1}^1 \sharp \cdots$ such that

$$\forall n\ (\sharp j_0^n \text{¢} \cdots \text{¢} j_{m_n}^n \sharp\ \sqsubset p\ \&\ \forall s \le m_n(\mathrm{rd}(j_s^n) < 2^{-n})).$$

Then, $\kappa'_X(q) = G$, hence $\kappa(p) = \kappa'(f_M(p))$. This means that $\kappa \le \kappa'$.

Let $\kappa'(p) = G$. Then p enumerates a Cauchy sequence of finite basic open covers of G which intersects G and has the limit G in the sense of Hausdorff metric. Suppose, without lost of generality, that $p = \sharp i_0^0 \text{¢} \cdots \text{¢} i_{m_0}^0 \sharp i_0^1 \text{¢} \cdots \text{¢} i_{m_1}^1 \sharp \cdots$ and $\forall n\ (d_H(\bigcup_{s\le m_n} U_{i_s^n}\ ,\ G) < 2^{-n})$. Construct a Turing machine M such that, for any input p, M will enumerate all $j_0 \text{¢} j_1 \text{¢} \cdots \text{¢} j_m$ for $m \in \omega$ which satisfy that,

$$\exists n \in \omega\Big(\bigcup\nolimits_{s\le m_n} U_{i_s^n} \subseteq \bigcup\nolimits_{t\le m} U_{j_t}\ \&\ \forall t \le m\ (\bigcup\nolimits_{s\le m_n} \tilde{U}_{i_s^n}^{(n)} \bigcap U_{j_t} \ne \emptyset)\Big),$$

where $\tilde{U}_{\langle i,j\rangle}^{(n)} = \{x \in X : d(\alpha(i), x) < \nu_Q(j) - 2^{-n}\}$. Note that $\tilde{U}_{i_s^n}^{(n)} \subseteq G$ for any $s \le m_n$, because $d_H(\bigcup_{s\le m_n} U_{i_s^n}\ ,\ G) < 2^{-n}$ and G is compact. Then it is not difficult to see that $M(p)$ enumerates all finite basic open covers of G which intersects G, hence $\kappa(f_M(p)) = G$. Therefore, f_M witenesses the reduction that $\kappa' \le \kappa$. $\square$

Classically, a metric space X is called locally compact if, for any $x \in X$, there is a neighbourhood B_x of x such that its closure $\overline{B_x}$ is compact. If such neighbourhood can be obtained effectively in x, then we will call it effectively locally compact. Now we define this concept precisely.

Definition 3. A computable metric space (M, d, A, α) is called *effectively locally compact* if there is a (ρ_X, ρ_R)-computable function $\gamma_X : X \to \mathbb{R}$ such that $\overline{B}(x, \gamma_X(x))$ is compact for any $x \in X$.

Obviously, the space $(\mathbb{R}, d_R, \mathbb{Q}, \alpha_Q)$ is effectively locally compact.

As end of this section, we show a simple property about computable metric space that, the metric d of any computable metric space is a computable function.

Lemma 4. *Let (X, d_X, A, α) be a computable metric space. Then the metric d is a (ρ_X, ρ_X, ρ_R)-computable function.*

Proof Given $x, y \in X$ with $\rho_X(p) = x$, $\rho_X(q) = y$. By the basic properties of the metric, we have $|d_X(x,y) - d_X(\alpha p(n), \alpha q(n))| \leq d_X(x, \alpha p(n)) + d_X(y, \alpha q(n)) < 2^{-n+1}$, and hence

$$\lim_{n\to\infty} d_X(\alpha p(n), \alpha q(n)) = d_X(x,y), \text{ and}$$
$$d_X(\alpha p(n), \alpha q(n)) - 2^{-n+1} < d_X(x,y) < d_X(\alpha p(n), \alpha q(n)) + 2^{-n+1}.$$

Define now inductively two sequences $\{a_n\}$ and $\{b_n\}$ of natural numbers by,

$$\begin{aligned} a_0 &= \mu m(\nu_Q(m) < d_X(\alpha p(0), \alpha q(0)) - 2); \\ b_0 &= \mu m(\nu_Q(m) > d_X(\alpha p(0), \alpha q(0)) + 2); \\ a_{n+1} &= \mu m(\nu_Q(a_n) < \nu_N(m) < d_X(\alpha p(n), \alpha q(n)) - 2^{-n+1}); \\ b_{n+1} &= \mu m(\nu_Q(b_n) > \nu_N(m) > d_X(\alpha p(n), \alpha q(n)) + 2^{-n+1}). \end{aligned}$$

By the recursive enumerability of D in the definition , it is easy to see that the sequences a_n and b_n are recursive in p and q. That is, there is a Turing machine M such that $f_M(p,q) = \sharp a_0 ¢ b_0 \sharp a_1 ¢ b_1 \sharp \cdots$. Let $r_n^1 = \nu_Q(a_n)$ and $r_n^2 = \nu_Q(b_n)$. Then the sequences r_n^1 and r_n^2 satisfy

$$\forall n(r_n^1 < r_{n+1}^1 < d_X(x,y) < r_{n+1}^2 < r_n^2) \;\&\; \lim_{n\to\infty}(r_n^2 - r_n^1) = 0.$$

This means that $d_X(x,y) = \rho_R(f_M(p,q))$. Hence the metric function d_X is (ρ_X, ρ_X, ρ_R)-computable. $\square$

3 Global Modulus of Continuity

In this section we will prove our main theorem. The proof is a direct construction of an algorithm which determines an appropriate value of δ, for any triple (f, x, ϵ), such that (1) is satisfied. The crucial idea is as follows: given any continuous function $f : X \to Y$, and element $x \in X$, consider at first the function $h : \mathbb{R}^+ \to \mathbb{R}^+$ defined for any $\delta \in \mathbb{R}^+$ by

$$\begin{aligned} h(\delta) &:= \min\{\epsilon \in \mathbb{R}^+ : f(\overline{B}(x,\delta)) \subseteq \overline{B}(f(x), \epsilon)\} \\ &= \max\{d_X(f(x), f(x')) : d_Y(x,x') \leq \delta \;\&\; x' \in X\}. \end{aligned}$$

Here we use the closed ball $\overline{B}$ instead of open ball B because a computable function f maps compact sets to compact sets and this does not work for open sets. h is nondecreasing and it is easy to see that, if $\epsilon = h(\delta_0)$, then δ_0 satisfies (1). Thus it suffices to define $\hat{\delta}$ as the reverse function of h. Unforturnately, h^{-1} does not necessarily exist, if h is not strictly increasing. To avoid this bad situation, let simply $g(\delta) = h(\delta) + \delta$ and $\delta_1 = g^{-1}(\epsilon)$. Then δ_1 satisfies (1) too. The following technical details make sure that the function g^{-1} is computable and, furthermore, depends effectively on the function $f \in C(X, Y)$ and the element $x \in X$.

Next lemma shows that the computable function maps effectively and uniformly any compact set of a computable metric space to a compact set.

Lemma 5. *Let X, Y be computable metric spaces and K_X the set of all compact subsets of X. Then the evaluation function $F : C(X, Y) \times K_X \to K_X$ defined by $F(f, B) := f(B)$ is $(\delta_{XY}, \kappa_X, \kappa_Y)$-computable.*

Proof Suppose that $f = \delta_{XY}(p)$ and $B = \kappa_X(q)$. Then p enumerates a sequence of $i\text{¢}j$'s which satisfy $f(\overline{U_i}) \subseteq U_j$ and such that

$$\forall x \in X \ \forall \epsilon > 0 \ \exists \langle i, j\rangle \ (\sharp i\text{¢}j\sharp \sqsubset p \ \& \ x \in U_i \ \& \ \mathrm{rd}(j) < \epsilon)$$

and q enumerates all strings $i_0\text{¢}i_1\text{¢} \cdots \text{¢}i_n$ for $n \in \omega$ such that $\{U_{i_0}, U_{i_1}, \ldots, U_{i_n}\}$ is a finite basic open covers of B which intersects B.

Construct a Turing machine M as following: input $(p, q) \in \Sigma^\omega \times \Sigma^\omega$, $M(p, q)$ will enumerate all strings $j_0\text{¢}j_1\text{¢} \cdots \text{¢}j_n$ for $n \in \omega$ which satisfy that: there are $m \in \omega$, $i_0, i_1, \ldots i_m$ and $k_0, k_1, \ldots k_m$ such that

1. $\sharp i_0\text{¢}i_1\text{¢} \cdots \text{¢}i_m\sharp \sqsubset q$;
2. $\forall s \leq m \ (\sharp i_s\text{¢}k_s\sharp \sqsubset p)$;
3. $\forall s \leq m \ \exists t \leq n \ (U_{k_s} \subset U_{j_t})$; and
4. $\forall t \leq n \ \exists s \leq m \ (U_{k_s} \subset U_{j_t})$.

Now, if $\{j_0, j_1, \ldots, j_n\}$ satisfies $(1 - 4)$ above for $m \in \omega$ and the finite sets $\{i_0, i_1, \ldots i_m\}$ and $\{k_0, k_1, \ldots k_m\}$, then $\{U_{i_t} : t \leq n\}$ is a finite basic open cover of the compact set B which intersects B, hence $\{U_{j_s} : s \leq n\}$ is a finite basic open cover of $f(B)$ becauce $\forall s \leq m \ \exists t \leq n \ (f(\overline{U}_{i_s}) \subset U_{k_s} \subset U_{j_t})$. Because $\forall t \leq n \ \exists s \leq m \ (U_{k_s} \subset U_{j_t})$ we have also that $\forall t \leq n \ (U_{j_t} \bigcap f(B) \neq \emptyset)$. This means that $\{U_{j_s} : s \leq n\}$ is a finite basic open cover of $f(B)$ which intersects $f(B)$.

On the other hand, suppose that $\{U_{j_t} : t \leq n\}$ is a finite basic open cover of $f(B)$ which intersects $f(B)$. For any $x \in A$, there is a $t \leq n$ such that $f(x) \in U_{j_t}$. By the continuity of f and the definition of δ_{XY}, there are $i, k \in \omega$ such that $x \in U_i \ \& \ f(U_i) \subset U_k \subset U_{j_t}$. This means that

$$\Gamma = \{U_i : i \in \omega \ \& \ \exists k \in \omega \ \exists t \leq n \ (\sharp i\text{¢}k\sharp \sqsubset p \ \& \ U_k \subset U_{j_t})\}$$

is a basic open cover of B. By the compactness of B, there is a finite subcover $\{U_{i_s}\}_{s \leq m} \subseteq \Gamma$ for some $m \in \omega$ which intersects B. That is, the string $j_0\text{¢}j_1\text{¢} \cdots \text{¢}j_n$ satisfies above conditions $1 - 4$ for some $i_0, \ldots i_m$ and $k_0, \ldots k_m$, hence will be enumerated finally by $M(p, q)$. That is, $M(p, q)$ enumerates all finite basic open cover of $f(B)$ which intersects $f(B)$.

Thus $\kappa_Y(f_M(p, q)) = f(B) = F(f, B)$, that is, the evaluation function F is $(\delta_{XY}, \kappa_X, \kappa_Y)$-computable. $\square$

Lemma 6. *Let* (X, d, A, α) *be a computable metric space. Then the distance function* $\tilde{d} : X \times K_X \to \mathbb{R}$ *defined by* $\tilde{d}(x, B) = \min\{d(x, y) : y \in B\}$ *is* $(\rho_X, \kappa_X, \rho_R)$*-computable.*

Proof Suppose that $x = \rho_X(p_1)$ with $p_1 = \sharp i_0 \sharp i_1 \sharp i_2 \sharp \cdots$ and $B = \kappa_X(p_2)$. From p_2, we can effectively construct a sequence p_2' such that

$$p_2' = \sharp i_0^0 ¢ \cdots ¢ i_{m_0}^0 \sharp i_0^1 ¢ \cdots ¢ i_{m_1}^1 \sharp \cdots \ \& \\ \forall n (\sharp i_0^n ¢ \cdots ¢ i_{m_n}^n \sharp \sqsubset p_2 \ \& \ \forall j \leq m_n (\mathrm{rd}(i_j^n) < 2^{-(n+1)}).$$

Furthermore we can construct effectively two sequences $\{s_n^i\}_{n \in \omega}$ (for $i = 0, 1$) of natureal numbers which satisfy, for any n,

$$\nu_Q(s_n^1) < \nu_Q(s_{n+1}^1) < \min_{j \leq m_n} d(\alpha\pi_1(i_n), \alpha\pi_1(i_j^n)) - 2^{-n} \ \& \\ \min_{j \leq m_n} d(\alpha\pi_1(i_n), \alpha\pi_1(i_j^n)) + 2^{-n} < \nu_Q(s_{n+1}^2) < \nu_Q(s_n^2).$$

Let $r_n^i = \nu_Q(s_n^i)$ for all $n \in \omega$ and $i = 1, 2$. It is not difficult to see that

$$\forall n (r_n^1 < r_{n+1}^1 < \tilde{d}(x, B) < r_{n+1}^2 < r_n^2) \ \& \ \lim_{n \to \infty} (r_n^2 - r_n^1) = 0.$$

Thus we can construct a Turing machine M such that $f_M(p_1, p_2) = \sharp s_0^1 ¢ s_0^2 \sharp s_1^1 ¢ s_1^2 \sharp s_2^1 ¢ s_2^2 \sharp \cdots$. So $\tilde{d}(x, B) = \rho_R(f_M(p_1, p_2))$. Hence $\tilde{d}$ is a $(\rho_X, \kappa_X, \rho_R)$-computable function. $\square$

We are now able to formulate our main theorem precisely and to prove it.

Theorem 7. *Let* (Y, d_Y, A_Y, α_Y) *be a computable metric space,* (X, d_X, A_X, α_X) *be an effectively locally compact computable metric space, let* $C(X, Y)$ *be the set of all continuous functions from* X *to* Y*. Then there is a* $(\delta_{XY}, \rho_X, \rho_R, \rho_R)$*-computable function* $\hat{\delta} : C(X, Y) \times X \times \mathbb{R}^+ \to \mathbb{R}^+$ *such that, for any* $(f, x, \epsilon) \in C(X, Y) \times X \times \mathbb{R}^+$*,*

$$\forall y \in X \ (d_X(x, y) < \hat{\delta}(f, x, \epsilon) \Longrightarrow d_Y(f(x), f(y)) < \epsilon). \tag{5}$$

That is, there is a $(\delta_{XY}, \rho_X, \rho_R, \rho_R)$*-computable global modulus of continuity of* $C(X, Y)$*.*

Proof Let γ_X be the (ρ_X, ρ_R)-computable function which witnesses the effectively local compactness of X. Define at first a function $\beta : X \times \mathbb{R} \to K_X$ by

$$\beta(x, \delta) = \begin{cases} \overline{B}(x, \gamma_X(x)) & \text{if } \delta > \gamma_X(x), \\ \overline{B}(x, \delta) & \text{otherwise.} \end{cases}$$

Then $\beta(x, \delta)$ is a compact subset of X for any $x \in X$ and $\delta \in \mathbb{R}$. Given any $x = \rho_X(p)$ and $\delta = \rho_R(q)$, the sequences $\{\alpha_X p(n)\}_{n \in \omega}$ and $\{\nu_Q q(n)\}_{n \in \omega}$ are two Cauchy sequences of the corresponding spaces with the limits x and δ, respectively. Because γ_X is (ρ_X, ρ_R)-computable, there is a Turing machine N such that $\rho_X(f_N(p)) = \gamma_X(x)$. Let $p_1 = f_N(p)$. Without lost of generality, we can assume that,

1. $d_X(x, \alpha_X(p(n))) < 2^{-(n+3)}$;
2. $|\delta - \nu_Q(q(n))| < 2^{-(n+3)}$; and
3. $|\gamma_X(x) - \nu_Q(p_1(n))| < 2^{-(n+3)}$.

Let $x_n = \alpha_X p(n)$ and $r_n = \min\{\nu_Q(q(n)), \nu_Q(p_1(n))\} + 2^{-(n+2)}$. Then $\beta(x, \delta) \subseteq B(x_n, r_n)$ for all $n \in \omega$.

Define $i_n = p(n)$ and $j_n = \mu j\ (r_n < \nu_Q(j) < r_n + 2^{-(n+3)})$. It is not difficult to see that $\beta(x, \delta) \subseteq U_{\langle i_n, j_n \rangle}$ and

$$\begin{aligned} d_H(U_{\langle i_n, j_n \rangle}, \beta(x, \delta)) &< d_X(x, \alpha_X(i_n)) + |r_n - \min\{\gamma_X(x), \delta\}| + |\nu_Q(j_n) - r_n| \\ &< 2^{-n}, \end{aligned}$$

for any $n \in \omega$. This means that $\{U_{\langle i_n, j_n \rangle}\}$ is a basic open cover of $\beta(x, \delta)$ satisfying that $d_H(U_{\langle i_n, j_n \rangle}, \beta(x, \delta)) < 2^{-n}$. Now we can construct a Turing machine M such that $f_M(p, q)$ enumerates the sequence $q_1 := \sharp \langle i_0, j_0 \rangle \sharp \langle i_1, j_1 \rangle \sharp \langle i_2, j_2 \rangle \sharp \cdots$. Then we have $\kappa'_X(f_M(p, q)) = \beta(x, \delta)$ for any $(p, q) \in \mathrm{dom}(\rho_X) \times \mathrm{dom}(\rho_R)$. That is, β is $(\rho_X, \rho_R, \kappa'_X)$-computable. By Proposition 2, β is also $(\rho_X, \rho_R, \kappa_X)$-computable.

Define a function $h : C(X, Y) \times X \times \mathbb{R}^+ \to \mathbb{R}^+$ by

$$h(f, x, \delta) = \max\{d_Y(f(x), y) : y \in f\beta(x, \delta)\}.$$

By the $(\rho_X, \rho_R, \kappa_X)$-computability of β and Lemma 5, function F defined by $F(f, x, \delta) = f\beta(x, \delta)$ is $(\delta_{XY}, \rho_X, \rho_R, \kappa_Y)$-computable. Hence, by Lemma 6, h is a $(\delta_{XY}, \rho_X, \rho_R, \rho_R)$-computable function.

Note that $h(f, x, \delta)$ is also nondecreasing on δ. So $g(f, x, \delta) := h(f, x, \delta) + \delta$ is an on δ strictly increasing $(\delta_{XY}, \rho_X, \rho_R, \rho_R)$-computable function. Then we can define a $(\delta_{XY}, \rho_X, \rho_R, \rho_R)$-computable function $\hat{\delta}_1 : C(X, Y) \times X \times \mathbb{R}^+ \to \mathbb{R}^+$ by $\hat{\delta}_1(f, x, \epsilon) := \iota\delta(\epsilon = g(f, x, \delta))$, where "$\iota$" is the "minimal value" operator.

Finally, we define $\hat{\delta} : C(X, Y) \times X \times \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\hat{\delta}(f, x, \epsilon) = \begin{cases} \hat{\delta}_1(f, x, \epsilon/2) & \text{if } \hat{\delta}_1(f, x, \epsilon/2) \leq \gamma_X(x), \\ \gamma_X(x) & \text{otherwise.} \end{cases}$$

Similar to the classical proofs of the results on the computability of inverse function and patching function, (see e.g., [7]), it is easy to see that $\hat{\delta}$ is $(\delta_{XY}, \rho_X, \rho_R, \rho_R)$-computable. By the definitions of h and g, we have, for any $\delta \leq \gamma_X(x)$, that

$$f(\overline{B}(x, \delta)) \subseteq \overline{B}(f(x), h(f, x, \delta)) \subseteq \overline{B}(f(x), g(f, x, \delta)).$$

Because $\hat{\delta}(f, x, \epsilon) \leq \gamma_X(x)$ and $g(f, x, \hat{\delta}_1(f, x, \epsilon)) = \epsilon$ for any $\epsilon \geq 0$, it follows that

$$\forall \epsilon \geq 0 (f(\overline{B}(x, \hat{\delta}(f, x, \epsilon)) \subseteq f(\overline{B}(x, \hat{\delta}_1(f, x, \epsilon/2)) \subseteq \overline{B}(f(x), \epsilon/2)).$$

That is

$$\forall x' \in X(d_X(x, x') < \hat{\delta}(f, x, \epsilon) \Longrightarrow d_Y(f(x), f(x')) < \epsilon).$$

So the function $\hat{\delta}$ satisfies the theorem. □

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