Weak computability and representation of reals

Xizhong Zheng*1,2 and Robert Rettinger**3

1 Theoretische Informatik, BTU Cottbus, 03044 Cottbus, Germany
2 Department of Computer Science, Jiangsu University, Zhenjiang 212013, China
3 Theoretische Informatik II, FernUniversit¨at Hagen, 58084 Hagen, Germany

Received 1 December 2003, revised 19 April 2004, accepted 4 March 2004
Published online 16 August 2004

Key words Computable real, weak computable real, Dedekind cut, binary expansion, Cauchy sequence, weakly computable real, Ershov’s hierarchy.

MSC (2000) 03F60, 03D55

The computability of reals was introduced by Alan Turing [20] by means of decimal representations. But the equivalent notion can also be introduced accordingly if the binary expansion, Dedekind cut or Cauchy sequence representations are considered instead. In other words, the computability of reals is independent of their representations. However, as it is shown by Specker [19] and Ko [9], the primitive recursiveness and polynomial time computability of the reals do depend on the representation. In this paper, we explore how the weak computability of reals depends on the representation. To this end, we introduce three notions of weak computability in a way similar to the Ershov’s hierarchy of $\Delta^0_2$-sets of natural numbers based on the binary expansion, Dedekind cut and Cauchy sequence, respectively. This leads to a series of classes of reals with different levels of computability. We investigate systematically questions as on which level these notions are equivalent. We also compare them with other known classes of reals like c.e. and d-c.e. reals.

1 Introduction

It is well known that a real can be represented by Dedekind cuts, Cauchy sequences, binary or decimal expansions. In classical mathematics, it makes no difference which representation is used. Based on the decimal expansion, Alan Turing [20] explored already computability of the reals. According to Turing, the computable numbers may be described briefly as the real numbers whose expressions as a decimal are calculable by finite means ([20, p. 230]. To give a precise definition of the “finite means”, Turing described a finite machine model which is now called Turing machine. By coding the natural numbers with finite strings of an alphabet, a Turing machine can compute a number-theoretical function. Such kind of functions can be naturally called computable. In addition, a set $A \subseteq \mathbb{N}$ is called computable if its characteristic function is computable. Thus, Turing’s definition can be rephrased as follows: a real $x \in [0; 1]$ is computable if there is a computable function $f: \mathbb{N} \rightarrow \{0, 1, \ldots, 9\}$ such that $x = \sum_{i \in \mathbb{N}} f(i) \cdot 10^{-i}$. Now it is natural to ask, when we define computability of reals based on other representations, are they equivalent to Turing’s original definition? The positive answer of this question was first observed by Robinson [15] and proved more formally latter by Myhill [10] and Rice [14].

Theorem 1.1 (Robinson [15], Myhill [10] and Rice [14]) For any real $x \in [0; 1]$, the following are equivalent.

1. $x$ is computable.
2. The Dedekind cut $L_x := \{r \in \mathbb{Q} : r < x\}$ of $x$ is a computable set.
3. There is a computable set $A \subseteq \mathbb{N}$ such that $x = x_A := \sum_{i \in A} 2^{-(i+1)}$.

* Corresponding author: e-mail: zheng@informatik.tu-cottbus.de
** e-mail: robert.rettinger@fernuni-hagen.de

1) In this paper we consider only the reals of the unit interval $[0; 1]$. For other reals $y$, there are an $n \in \mathbb{N}$ and an $x \in [0; 1]$ such that $y := x \pm n$. $y$ and $x$ are regarded as being of the same computability.
4. There is a computable sequence \((x_s)\) of rational numbers which converges to \(x\) effectively in the sense that

\[
(\forall s, t \in \mathbb{N}) \ (t \geq s \Rightarrow |x_s - x_t| \leq 2^{-s}).
\]

It is worth noting that the extra condition (1) above is essential for the computability of a real \(x\), because Specker [19] showed that there is an increasing computable sequence of rational numbers which converges to a non-computable real. In other words, not every sequence converges effectively!

Furthermore, as observed by Specker [19], the equivalence in Theorem 1.1 does not hold if the primitive recursiveness is considered instead of computability. More precisely, let \(\mathcal{R}_1\) be the class of all limits of primitive recursive sequences of rational numbers which converge primitive recursively (i.e., with a primitive convergence modulus), \(\mathcal{R}_2\) be the class of all reals of primitive recursive binary expansions and \(\mathcal{R}_3\) include all reals of primitive recursive Dedekind cuts. Then it is shown in [19] that \(\mathcal{R}_3 \not\subseteq \mathcal{R}_2 \not\subseteq \mathcal{R}_1\). Ko [9] showed that the polynomial time computability of reals depends on their representations too. Let \(\mathcal{P}_c\) be the class of limits of all polynomial time computable sequences of dyadic rational numbers which converge effectively, \(\mathcal{P}_D\) contain all reals of polynomial time computable Dedekind cuts and \(\mathcal{P}_B\) be the class of reals whose binary expansions are polynomial time computable (with the input \(n\) written in unary notation). Ko [9] shows that \(\mathcal{P}_D = \mathcal{P}_B \subseteq \mathcal{P}_C\) and \(\mathcal{P}_C\) is a real closed field while \(\mathcal{P}_D\) is not closed under addition and subtraction. In [9], the dyadic rational numbers \(\mathcal{D} := \bigcup_{n \in \mathbb{N}} \mathbb{D}_n\) for \(\mathbb{D}_n := \{m : 2^{-n} : m \in \mathbb{Z}\}\) instead of \(\mathbb{Q}\) is used as base set. For the complexity discussion, the class \(\mathcal{D}\) seems more natural and easier to use. But for computability it makes no essential difference and we use both \(\mathcal{D}\) and \(\mathbb{Q}\) in this paper.

In this paper, we will investigate the question how different notions of weak computability of the reals depend on their representations. To this end, of course, we have to introduce the precise notion of “weak computability” first. Several weak computability notions of the reals appeared already in literature. For example, a real \(x\) is called left (right) computable if there exists an increasing (decreasing) computable sequence of rational numbers which converges to \(x\). The left computable reals are also called c.e. because their left Dedekind cuts are c.e. sets. Left and right computable reals are called semi-computable. If \(x\) is the difference of two left computable reals, then \(x\) is called weakly computable or d-c. e. According to Ambos-Spies, Weihrauch and Zheng [1], \(x\) is weakly computable iff there is a computable sequence \((x_s)\) of rational numbers which converges to \(x\) weakly effectively, in the sense that \(\sum_{s\in \mathbb{N}}|x_s - x_{s+1}| \leq c\) for a constant \(c\). More generally, if \(x\) is simply the limit of a computable sequence of rational numbers, then \(x\) is called computably approximable. The classes of computable, left computable, right computable, semi-computable, weakly computable and computably approximable reals are denoted by \(\mathcal{EC}\), \(\mathcal{LC}\), \(\mathcal{RC}\), \(\mathcal{SC}\), \(\mathcal{WC}\) and \(\mathcal{CA}\), respectively. The relationship among these classes is

\[
\mathcal{EC} = \mathcal{LC} \cap \mathcal{RC} \subseteq \mathcal{SC} = \mathcal{LC} \cup \mathcal{RC} \subseteq \mathcal{WC} \subseteq \mathcal{CA}
\]

as showed in [1].

As observed by Jockusch (cf. [17]), if \(A\) is a non-computable c.e. set, then the real \(x_{A@\overline{A}}\) is c.e. but its binary expansion \(A \oplus \overline{A} := \{2n : n \in A\} \cup \{2n + 1 : n \notin A\}\) is not a c.e. set. In other words, the computable enumerability of a real and the computable enumerability of its binary expansion is not equivalent. Furthermore, Soare [17] showed that the binary expansion set \(A\) of a c.e. real \(x_A\) may be very far from being c.e., and may even be cohesive, where a set \(A\) is called cohesive if there is no c.e. set \(W\) such that \(W \cap A\) and \(\overline{W} \cap A\) are both infinite. We will see that the reason why such kind of phenomena appears is that the weak computable reals based on Dedekind cuts collapse up to certain level. To this end, we explore the notion of weak computability of reals more systematically.

Notice that, weak computability deals mainly with non-computable objects and non-computable objects are typically classified in computability theory into equivalent classes or so-called degrees by various reducibilities (see e.g. [18, 11, 12]). One of the most important reducibility is the Turing reducibility and the corresponding degree is called Turing degree. This approach can be easily transferred to the reals by mapping each set \(A \subseteq \mathbb{N}\) to a real \(x_A := \sum_{i \in A} 2^{-i+1}\) and defining the Turing reducibility of reals by \(x_A \leq_T x_B\) if and only if \(A \leq_T B\). This definition is robust as shown in [23, 2] in the sense that it does not depend on the representation of reals. That is, if we define Turing reducibility based on other representations of the reals, then we obtain an equivalent reducibility. The advantage of this approach is that the techniques and results from well developed computability theory can be applied straightforwardly. For example, Ho [8] shows that a real \(x\) is \(\Delta^0_2\) or, equivalently, is Turing

© 2004 WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim
reducible to the halting problem $K$, iff there is a computable sequence of rational numbers which converges to $x$. This is a reprint of Shoenfield’s Limit Lemma ([16]) in computability theory which says that $A \leq_T K$ iff $A$ is a limit of a computable sequence of subsets of natural numbers. On the degree of d-c.e. real, the first author showed in [22] that there exists a d-c.e. real whose Turing degree is not $\omega$-c.e. Recently, Downey, Wu and Zheng [2] showed that there exists a $\Delta^0_2$-degree which does not contain any d-c.e. real.

Turing reducibility is a very useful tool to classify the objects according to their non-computability level. However, this classification seems not fine enough and it does not reveal too much information about weak computability. For the reals, one of the natural requirement of “weak computability” is that it should be approximated at least by a computable sequence of rational numbers. That is, we can restrict ourself to the $\Delta^0_2$-reals. In this case, we can apply the well-behaved hierarchy of Ershov ([7]) for $\Delta^0_2$-subsets of natural numbers. Again, this hierarchy can be transferred to reals via their binary expansions straightforwardly. More precisely, we call a real $x_A$ h-binary computable if the set $A$ is h-c.e. in the Ershov hierarchy. Similarly, after extending Ershov’s Hierarchy to subsets of rational numbers, we can call a real $x$ h-Dedekind computable if its (left) Dedekind cut is an h-c.e. set of rational numbers. For the Cauchy representation, a classification similar to Ershov’s can be introduced too. In this case, we count the number of the “big jumps” of the sequence which converges to the real. According to Theorem 1.1.4, $x$ is computable if there is a computable sequence $(x_s)$ of rational numbers which converges to $x$ and the sequence $(x_s)$ makes no big jumps in the sense of (1). However, if up to $h(n)$ (non-overlapped) “big jumps” of the distance around $2^{-n}$ are allowed in the sequence, then $x$ is called $h$-Cauchy computable. In this way, three kinds of $h$-computability of reals are naturally introduced.

In this paper, we will investigate these notions in detail and compare them with each other and also with other known classes of reals of weak computability mentioned above. We show that, for any constant $k \in \mathbb{N}$, the classes of $k$-Dedekind computable reals collapse to the second level, the class of 2-Dedekind computable reals which is equal to the class of semi-computable reals. However, the hierarchy theorem for binary and Cauchy computability holds. But no class of the $k$-binary and $k$-Dedekind computable reals is comparable to the class of semi-computable reals for $k \geq 2$. Very interestingly, for the $\omega$-computability (i.e., the union of $h$-computability of all computable functions $h$), we obtain exactly the same relationship among the three computability versions as one of polynomial time computability. Namely, the $\omega$-binary computability is equivalent to the $\omega$-Dedekind computability and they are not closed under addition and subtraction. The $\omega$-Cauchy computability is closed under the arithmetical operations and is strictly weaker than $\omega$-binary and $\omega$-Dedekind computability.

This paper is organized as follows. The basic definitions are given in Section 2 and the Sections 3 to 5 contribute to the binary computability, Dedekind computability and Cauchy computability, respectively.

2 Basic definitions

In this section, we recall the definition of Ershov’s hierarchy of $\Delta^0_2$-subsets of natural numbers and give the precise definitions of binary, Dedekind and Cauchy computability.

Notice that, if a set $A \subseteq \mathbb{N}$ is computable, then there is an algorithm which tells us whether a natural number $n$ belongs to $A$ or not. In this case, corrections are not allowed. However, if we allow the algorithm to change its mind for the membership of $n$ to $A$ but only from negative to positive, then the corresponding set $A$ is a c.e. set. In other words, the algorithm may claim $n \notin A$ wrongly at some stage and correct its claim to $n \in A$ at a later stage. In general, given a function $h : \mathbb{N} \rightarrow \mathbb{N}$, if the algorithm is allowed to change the answer to the question “$n \in A$?” at most $h(n)$ times for any $n \in \mathbb{N}$, then the corresponding set $A$ is called $h$-computably enumerable (h-c.e. for short). This leads to the well-known hierarchy of Ershov [7, 6]. For the precise definition, let’s introduce some useful notions at first. For any finite set $A := \{x_1 < x_2 < \cdots < x_k\}$ of natural numbers, the natural number $i := 2^{x_1} + 2^{x_2} + \cdots + 2^{x_k}$ is called the canonical index of $A$. The set with canonical index $i$ is denoted by $D_i$. A sequence $(A_s)$ of finite subsets of $\mathbb{N}$ is called computable if there is a computable function $g : \mathbb{N} \rightarrow \mathbb{N}$ such that $A_s = D_{g(s)}$ for all $s \in \mathbb{N}$. Similarly, we can introduce the canonical index for subsets of dyadic rational numbers. Let $\sigma : \mathbb{N} \rightarrow \mathbb{D}$ be a one-to-one effective coding of the dyadic numbers. The canonical index of a finite set $A \subseteq \mathbb{D}$ is defined as the canonical index of the set $A_{\sigma} := \sigma^{-1}(A) := \{n \in \mathbb{N} : \sigma(n) \in A\}$. In this paper, the subset $A \subseteq \mathbb{D}$ of canonical index $n$ is denoted by $V_n$. A sequence $(A_s)$ of finite subsets of dyadic numbers is called computable if there is a computable function $h$ such that $A_s = V_{h(s)}$ for all $s \in \mathbb{N}$.

© 2004 WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim
Definition 2.1 (Ershov [7], Epstein et al. [6]) Let \( h : \mathbb{N} \rightarrow \mathbb{N} \) be a function. A set \( A \subseteq \mathbb{N} \) is called \( h \)-computably enumerable (\( h \)-c.e. for short) if there is a computable sequence \( \langle A_n \rangle \) of finite subsets \( A_n \subseteq \mathbb{N} \) such that

1. \( A_0 = \emptyset \) and \( A = \lim_{n \to \infty} A_n \) (i.e., \( A = \bigcup_{i=0}^{\infty} \bigcap_{j=1}^{\infty} A_j \)), and
2. \( \langle \forall n \in \mathbb{N} \rangle \left( \{ \langle s : n \in A_n, \Delta A_{n+1} \rangle \} \leq h(n) \right) \), where \( \Delta A_{n+1} := (A \setminus B) \cup (B \setminus A) \).

In this case, the computable sequence \( \langle A_n \rangle \) is called a computable \( h \)-enumeration of \( A \). For \( k \in \mathbb{N} \), \( A \) is called \( k \)-c.e. if it is \( h \)-c.e. for the constant function \( h(n) \equiv k \) and \( A \) is \( \omega \)-c.e. if it is \( h \)-c.e. for some computable function \( h \). For convenience, computable sets are called \( 0 \)-c.e.

Theorem 2.2 (Hierarchy Theorem, Ershov [7] and Epstein [5]) Let \( f, g : \mathbb{N} \rightarrow \mathbb{N} \) be computable functions. If \( f(n) < g(n) \) holds for infinitely many \( n \), then there is a \( g \)-c.e. set which is not \( f \)-c.e. Thus, there is an \( \omega \)-c.e. set which is not \( k \)-c.e. for any \( k \in \mathbb{N} \); there is a \( (k + 1) \)-c.e. set which is not \( k \)-c.e. (for every \( k \in \mathbb{N} \)), and there is also a \( \Delta^0_2 \)-set which is not \( \omega \)-c.e.

The definition of \( h \)-c.e., \( k \)-c.e. and \( \omega \)-c.e. subsets of natural numbers can be transferred straightforwardly to subsets of dyadic rational numbers. Of course, \( h \) should have the type \( h : \mathbb{D} \rightarrow \mathbb{N} \) in this case. However, if it is clear from the context we often do not indicate this explicitly. Thus, we can easily introduce corresponding hierarchies for reals by means of binary or Dedekind representations of reals. However, if the reals are represented by Cauchy sequences, we cannot do that directly. Our suggestion here is to count the number of their jumps of certain size in this case. More precisely, we have the following definition.

Definition 2.3 Let \( n \) be a natural number and \( (x_n) \) be a sequence of reals which converges to \( x \).

1. An \( n \)-jump of \( (x_n) \) is an index pair \( (i, j) \) such that \( n < i < j \) and \( 2^{-n} \leq |x_i - x_j| < 2^{-n+1} \).
2. The \( n \)-divergence of \( (x_n) \) is the maximal number of non-overlapping \( n \)-jump pairs of \( (x_n) \), i.e., the maximal natural number \( m \) such that there exists a chain \( n < i_1 < j_1 \leq i_2 < j_2 \leq \cdots \leq i_m < j_m \) with \( 2^{-n} \leq |x_{i_t} - x_{j_t}| < 2^{-n+1} \) for \( t = 1, 2, \ldots, m \).
3. For \( h : \mathbb{N} \rightarrow \mathbb{N} \), the sequence \( (x_n) \) converges to \( x \) \( h \)-effectively if the \( n \)-divergence of \( (x_n) \) is bounded by \( h(n) \) for all \( n \in \mathbb{N} \).

Definition 2.4 Let \( x \in [0; 1] \) be a real and \( h \) a function.

1. \( x \) is \( h \)-binary computable (\( h \)-bEC for short) if there is an \( h \)-c.e. set \( A \subseteq \mathbb{N} \) such that \( x = x_A \).
2. \( x \) is \( h \)-Cauchy computable (\( h \)-cEC for short) if there is a computable sequence \( (x_n) \) of rational numbers which converges to \( x \) \( h \)-effectively.
3. \( x \) is \( h \)-Dedekind computable (\( h \)-dEC for short) if the Dedekind cut \( L_x := \{ r \in \mathbb{Q} : r < x \} \) is \( h \)-c.e.
4. For \( \delta \in \{ b, c, d \} \) and \( k \in \mathbb{N} \), \( x \) is called \( k \)-\( \delta \)EC if \( x \) is \( h \)-EC for the constant function \( h(n) \equiv k \) and \( x \) is called \( \omega \)-\( \delta \)EC if it is \( h \)-EC for a computable function \( h \).

The classes of all \( k \)-\( \delta \)EC, \( h \)-\( \delta \)EC and \( \omega \)-\( \delta \)EC reals are denoted by \( k \)-\( \delta \)EC, \( h \)-\( \delta \)EC and \( \omega \)-\( \delta \)EC, respectively, for \( \delta \in \{ b, c, d \} \) and \( k \in \mathbb{N} \). Besides, let \( \omega \)-\( \delta \)EC := \( \bigcup_{n \in \mathbb{N}} n \)-\( \delta \)EC. The following proposition follows directly from the definition.

Proposition 2.5 Let \( \delta \in \{ b, c, d \} \) and \( f, g \) be functions. Then the following hold.

1. \( 0 \)-\( \delta \)EC = EC and \( \omega \)-\( \delta \)EC \( \subseteq \) CA.
2. \( k \)-\( \delta \)EC \( \subseteq \) \( k + 1 \)-\( \delta \)EC \( \subseteq \) \( + \)-\( \delta \)EC \( \subseteq \) \( \omega \)-\( \delta \)EC, for any \( k \in \mathbb{N} \).
3. If \( f(n) \leq g(n) \) holds for almost all \( n \in \mathbb{N} \), then \( f \)-\( \delta \)EC \( \subseteq \) \( g \)-\( \delta \)EC.

3 Binary computability

In this section we discuss the binary computability of reals in details. Obviously, any \( 1 \)-binary computable real is left computable. However, we will show that, for \( k \geq 2 \), the classes \( k \)-bEC and \( + \)-bEC are not comparable with the class of semi-computable reals. On the other hand, the class \( \omega \)-bEC contains properly the class SC but is not comparable with WC.

Let’s look at the classes \( k \)-bEC for \( k \in \mathbb{N} \) at first. By the Hierarchy Theorem 2.2, the following proposition is straightforward.
Proposition 3.1 \( k \cdot b \text{EC} \subseteq (k+1) \cdot b \text{EC} \subseteq \ast \cdot b \text{EC} \subseteq \omega \cdot b \text{EC} \), for all \( k \in \mathbb{N} \).

To compare the binary computability with semi-computability, Jockusch (cf. [17]) observed that there is a left computable real which is not 1-binary computable, i.e., \( \text{LC} \not\subseteq 1 \cdot b \text{EC} \). This can even be extended to \( k \)-binary computable reals for any \( k \) (see the next theorem). On the other hand, in contrast with the simple fact \( 1 \cdot b \text{EC} \subseteq \text{LC} \), there exists a 2-binary computable real which is not semi-computable.

Theorem 3.2
1. \( \text{LC} \not\subseteq \ast \cdot b \text{EC} \).
2. \( 2 \cdot b \text{EC} \subseteq \text{SC} \).

Proof.
1. We construct a set \( A \subseteq \mathbb{N} \) such that \( x_A \) is left computable but \( A \) is not \( k \)-c.e. for any constant \( k \). That is, \( A \) has to satisfy, for all \( i, j \in \mathbb{N} \), the following requirements.

\[
R_{i,j} \text{ if } (D_{\varphi_i(s)})_s \text{ is a computable } j \text{-enumeration, then } A \neq \lim_{s \to \infty} D_{\varphi_i(s)},
\]

where \( (\varphi_i)_s \) is an effective enumeration of all computable partial functions \( \varphi_i : \subseteq \mathbb{N} \to \mathbb{N} \).

To satisfy the requirement \( R_{i,j} \) for \( e := \langle i, j \rangle \), we choose an \( n_e > j \). We put \( n_e \) into \( A \) as long as \( n_e \) is not in \( D_{\varphi_i(s)} \). If \( n_e \) enters \( D_{\varphi_i(s)} \) for some \( s \), then we take \( n_e \) out of \( A \). \( n_e \) may be put into \( A \) again if \( n_e \) leaves \( D_{\varphi_i(s)} \) for some \( t > s \), and so on. If the sequence \( (D_{\varphi_i(s)})_s \) is a computable \( j \)-enumeration, then \( n_e \) enters and leaves \( A \) at most \( j \) times. This guarantees that \( R_{i,j} \) can be satisfied eventually by at most \( j \) attacks of this strategy. In addition, to guarantee that the real \( x_A \) is left computable, we reserve an interval \( [m_e, n_e] \) of natural numbers with \( n_e - m_e > j \) exclusively for the requirement \( R_e \) and put a new element from this interval into \( A \) whenever \( n_e \) is taken out of \( A \). To satisfy all requirements simultaneously, a standard finite injury priority construction suffices. The details are omitted here.

2. As it is shown by Ambos-Spies, Weihrauch and Zheng ([1, Theorem 4.8]), if \( A, B \subseteq \mathbb{N} \) are two Turing incomparable c.e. sets, i.e., \( A \not\leq_T B \) and \( B \not\leq_T A \), then the real \( x_{A \oplus B} \) is not semi-computable, where the join \( A \oplus B := \{2n : n \in A\} \cup \{2n+1 : n \notin B\} \). On the other hand, for any c.e. sets \( A, B \), the join \( A \oplus B := (2A \cup (2\mathbb{N} + 1)) \setminus (2B + 1) \) is a obviously 2-c.e. set and hence \( x_{A \oplus B} \) is 2-bEC. That is, there exists a 2-bEC real which is not semi-computable.

An immediate consequence of Theorem 3.2 is that the class \( \text{SC} \) is not comparable with the classes \( k \cdot b \text{EC} \) for \( k \geq 2 \) and \( \ast \cdot b \text{EC} \). However, the next theorem shows that the class \( \text{WC} \) contains properly all classes \( k \cdot b \text{EC} \) for \( k \in \mathbb{N} \).

Theorem 3.3 \( \ast \cdot b \text{EC} \subseteq \text{WC} \).

Proof. For the inclusion, it suffices to prove that \( k \cdot b \text{EC} \subseteq \text{WC} \) by induction on \( k \) as follows. Assume by induction hypothesis that \( k \cdot b \text{EC} \subseteq \text{WC} \) for some \( k \in \mathbb{N} \). Let \( A \subseteq \mathbb{N} \) be a \( (k+1) \)-c.e. set. Then there exist a c.e. set \( B \) and a \( k \)-c.e. set \( C \) such that \( A = B \setminus C \). Obviously, the set \( B \cup C \) is \( k \)-c.e. too. Then, both \( x_{B \cup C} \) and \( x_C \) are \( k \cdot b \text{EC} \) and hence weakly computable by induction hypothesis, i.e., \( x_{B \cup C} \in \text{WC} \) and \( x_C \in \text{WC} \). Since the class \( \text{WC} \) is closed under subtraction and \( x_A = x_B(C) = x_{B \cup C} \setminus C = x_{B \cup C} - x_C \), \( x_A \) is weakly computable too. Therefore \( (k+1) \cdot b \text{EC} \subseteq \text{WC} \).

For the inequality, it is showed in [22] that there exists a weakly computable real \( x_A \) such that the set \( A \) is not \( \omega \cdot c. e. \). That is, there is some weakly computable real which is not \( \omega \cdot b \text{EC} \) and hence not \( \ast \cdot b \text{EC} \). Therefore \( \ast \cdot b \text{EC} \neq \text{WC} \).

Now we discuss the property of the class \( \omega \cdot b \text{EC} \). Different to the case of \( \ast \cdot b \text{EC} \), the class \( \omega \cdot b \text{EC} \) contains all semi-computable reals. However, it is not comparable with the class \( \text{WC} \).

Theorem 3.4
1. \( \text{SC} \subset \omega \cdot b \text{EC} \).
2. \( \text{WC} \nsubseteq \omega \cdot b \text{EC} \) and \( \omega \cdot b \text{EC} \nsubseteq \text{WC} \).
Proof.
1. As pointed out by Soare [17, p. 217], if a real $x_A$ is left computable, then the set $A$ is $\omega$-c.e. for the function $h(n) = 2^{n+1}$ for all $n$. On the other hand, by Theorem 2.2, there is a set $A$ which is $\omega$-c.e. but not $\omega$-c.e. This implies $\text{SC} \subseteq \omega$-b EC immediately.

2. In [22] the first author shows that there are $\text{c.e.}$ sets $A, B \subseteq \mathbb{N}$ such that the set $C \subseteq \mathbb{N}$ defined by $x_C := x_A - x_B$ is not Turing equivalent to any $\omega$-c.e. set. This means that the real $x_C$ is weakly computable but not $\omega$-b EC. That is, $\text{WC} \not\subseteq \omega$-b EC.

To prove $\omega$-b EC $\not\subseteq$ WC, let’s recall a result of Ambos-Spies, Weihrauch and Zheng [1] that if a real $x_A \oplus \emptyset$ is weakly computable and $h$ is defined by $h(n) := 2^{3n}$ for all $n$, then the set $A$ is $\omega$-c.e. By Ershov’s Hierarchy Theorem 2.2, we can choose an $\omega$-c.e. set $A$ which is not $\omega$-c.e. Then the set $B := A \oplus \emptyset$ is obviously also an $\omega$-c.e. set and hence $x_B$ is $\omega$-b EC. But $x_B$ is not weakly computable because $A$ is not $\omega$-c.e. \[
\]

An immediate consequence of Theorem 3.2.1 and Theorem 3.4 is that the class $\omega$-b EC is not closed under addition and subtraction, because WC is the arithmetical closure of SC. In fact, no other class of binary computable reals except $0$-b EC is closed under addition and subtraction.

Theorem 3.5 For $\delta \in \mathbb{N}^+ \cup \{*, \omega\}$, the class $\delta$-b EC is not closed under addition and subtraction.

Proof. In [22], the first author has constructed two $\text{c.e.}$ sets $A, B \subseteq \mathbb{N}$ such that if $x_C := x_A - x_B$, then the set $C$ is not $\omega$-c.e. This implies directly that the classes $\delta$-b EC are not closed under subtraction for $\delta \in \mathbb{N}^+ \cup \{*, \omega\}$. Notice that any co-$\text{c.e.}$ set is 2-c.e. and hence $-x_B$ is 2-b EC real if $B$ is a $\text{c.e.}$ set. Thus, above example implies that the classes $\delta$-b EC are not closed under addition if $\delta \geq 2$. For the class 1-b EC, let $A := \{2e + 1 : e \in \mathbb{N}\}$ and $B := \{2e + 1 : 2e + 1 \in W_e\}$ where $(W_e)$ is an effective enumeration of all $\text{c.e.}$ sets. Then $A$ and $B$ are $\text{c.e.}$ but $C$ is not $\text{c.e.}$ if $x_C = x_A + x_B$ because $C(2e + 1) \neq W_e(2e + 1)$ for all $e$. That is, 1-b EC is not closed under addition too.

Remark 3.6 In the literature, the 1-bEC reals are also called strongly c.e. (see e.g. [3]). Wu [21] extended this notion further to $n$-strongly c.e. reals which are the sum of up to $n$ strongly c.e. reals and he showed that they form a proper hierarchy. This implies also that 1-b EC is not closed under addition. However, for any $n \geq 2$, the class of $n$-strongly c.e. reals is not comparable to any classes $m$-b EC for $m \geq 2$.

4 Dedekind computability

We investigate Dedekind computability in this section. The situation now is quite different from the binary computability. The general hierarchy theorem does not hold any more. Actually, all classes $k$-dEC collapse to the second lever 2-dEC for $k \geq 2$. On the other hand, we will show that the $\omega$-Dedekind computability is equivalent to the $\omega$-binary computability.

First we show that the class $\ast$-dEC collapses to SC and hence the hierarchy theorem does not hold.

Lemma 4.1

1. $1$-dEC = LC and SC $\subseteq$ 2-dEC.
2. $\ast$-dEC = SC.
3. For all $k \in \mathbb{N}$, if $k \geq 2$, then $k$-dEC = SC.

Proof.
1. This follows directly from definition.
2. By 1., it suffices to prove that $\ast$-dEC $\subseteq$ SC. For any $x \in \ast$-dEC, let $k := \min\{n : x \in n$-dEC$\}$. If $k < 2$, then $x$ is left computable and we are done. Suppose now that $k \geq 2$. Notice that, the Dedekind cut $L_x := \{r \in \mathbb{D} : r < x\}$ of $x$ is a $k$-c.e. but not $(k - 1)$-c.e. set. Let $(A_x)$ be a computable $k$-enumeration of $L_x$. Then there are infinitely many $r \in \mathbb{D}$ such that $|\{s \in \mathbb{N} : r \in A_{s+1} \Delta A_s\}| = k$, where $A_{\delta} B := (A \setminus B) \cup (B \setminus A)$, otherwise, $L_x$ is $(k - 1)$-c.e. That is, the set $O_k := \{r \in \mathbb{D} : |\{s \in \mathbb{N} : r \in A_{s+1} \Delta A_s\}| = k\}$ is infinite. Obviously, $O_k$ is a c.e. set. If $k$ is even, then $r \notin L_x$ for any $r \in O_k$ (remember $A_0 = \emptyset$) and hence $x < r$.

Now we will show that $\inf O_k = x$ holds. Suppose by contradiction that $\inf O_k > x$. That is, there is a rational number $y$ such that $x < y < r$ for all $r \in O_k$. Then we can construct a computable $(k - 1)$-enumeration of $L_x$ by allowing any $r > y$ to enter $L_x$ at most $k/2 - 1$ times. This contradicts the hypothesis. Since $\inf O_k = x$, we can
choose a decreasing computable sequence \( (r_s) \) from \( O_k \) such that \( \lim r_s = x \) and hence \( x \) is right computable.

Similarly, if \( k \) is odd, then \( x \) is left computable.

3. follows immediately from 1. and 2.

As an immediate consequence of Lemma 4.1, Dedekind computability and binary computability are not equivalent for many levels except the trivial cases. We summarize the comparison between binary and Dedekind computability as follows.

**Theorem 4.2**

1. \( 0\text{-}b\ \text{EC} = 0\text{-}d\text{EC} \).
2. \( 1\text{-}b\ \text{EC} \not\subseteq 1\text{-}d\text{EC} \).
3. \( \delta\text{-}d\text{EC} \not\subseteq \delta\text{-}b\ \text{EC} \) and \( \delta\text{-}b\ \text{EC} \not\subseteq \delta\text{-}d\text{EC} \) if \( \delta \geq 2 \) or \( \delta = * \).

**Proof.**

1. By our convention, a set is 0-c.e. if and only of it is computable. Then the assertion follows directly from the definition and Theorem 1.1.

2. This was first observed by Jockusch (cf. [17]). Jockusch showed that there is a d-c.e. set \( A \) which is not c.e. such that \( x_A \) is left computable, i.e., \( x_A \in 1\text{-}d\text{EC} \setminus 1\text{-}b\text{EC} \). In addition, any real of a c.e. binary expansion is obviously left computable. This implies that \( 1\text{-}b\ \text{EC} \not\subseteq 1\text{-}d\text{EC} \).

3. Let \( 2 \geq \delta \) or \( \delta = * \). By Lemma 4.1 we have at first \( \delta\text{-}d\text{EC} = \text{SC} \). On the other hand, by Theorem 3.2, the class \( \text{SC} \) is not comparable with any \( \delta\text{-}b\ \text{EC} \). Therefore, we have \( \delta\text{-}d\text{EC} \not\subseteq \delta\text{-}b\ \text{EC} \) and \( \delta\text{-}b\ \text{EC} \not\subseteq \delta\text{-}d\text{EC} \). \( \square \)

Now let’s look at the \( \omega\text{-}Dedekind computability. By a construction we can show that \( \omega\text{-}d\text{EC} \) is incommensurable with \( \text{WC} \). More directly, we can get this by showing that the \( \omega\text{-}binary computability and \( \omega\text{-}Dedekind computability are actually equivalent.

**Theorem 4.3** \( \omega\text{-}b\ \text{EC} = \omega\text{-}d\text{EC} \).

**Proof.**

“\( \omega\text{-}b\ \text{EC} \subseteq \omega\text{-}d\text{EC} \)” Suppose that \( x \) is an \( \omega\text{-}b\text{EC} \) real. That is, there exists an \( \omega\text{-}c.\ e. \) set \( A \) such that \( x = x_A \). Let \( h \) be a computable function and \( (A_s) \) be a computable \( h\text{-}\text{enumeration of } A \). We are going to show that \( x_A \in \omega\text{-}d\text{EC} \), i.e., the left Dedekind cut of \( x_A \) is an \( \omega\text{-}c.\ e. \) set too.

To this end, we define a computable sequence \( (E_s) \) of finite subsets of dyadic numbers by \( E_s := \{ r \in \mathbb{D}_s : r < x_A \} \). Let’s identify a dyadic rational number \( r \) with a binary word denoted also by \( r \) in the sense that \( r = \sum_{i < l(r)} r(i) \cdot 2^{-(i+1)} \), where \( l(r) \) is the length of the word \( r \). Similarly, we identify a set \( A \) with its characteristic sequence, i.e., \( n \in A \) iff \( A(n) = 1 \) and \( n \notin A \) iff \( A(n) = 0 \). Then we have: \( r < x_A \) iff \( r <_L A \) iff \( (\forall s \leq t) (r \leq s, A_s) \) iff \( (\forall s \in E_s) \) iff \( r \in E := \lim_{s \to \infty} E_s \), where \( s <_L \) is the length-lexicographical ordering on binary words and sequences. Thus, the limit \( E := \lim_{s \to \infty} E_s \) is the left Dedekind cut of the real \( x_A \). On the other hand, for any dyadic rational number \( r \) and any natural numbers \( s \) and \( t \), if \( A_s \upharpoonright l(r) = A_t \upharpoonright l(r) \), then \( r \leq_L x_A \) iff \( r \leq_L x_A \), and hence \( r \in E \) iff \( r \in E \). Then, if the membership of \( r \) to \( E_s \) is changed, there must be a numbers \( n < l(r) \) which is put into or deleted from \( A_s \).

Since \( (A_s) \) is a computable \( h\text{-}\text{enumeration of } A \), the sequence \( (E_s) \) is a computable \( g\text{-}\text{enumeration of } E \), where \( g : \mathbb{D} \to \mathbb{N} \) is a computable function defined by \( g(r) := \sum_{i \leq l(r)} h(i) \). Thus, \( x \) is a \( g\text{-}\text{EC} \) and hence an \( \omega\text{-}d\text{EC} \) real.

“\( \omega\text{-}d\text{EC} \subseteq \omega\text{-}b\ \text{EC} \)” Suppose that \( x := x_A \) is \( \omega\text{-Dedekind computable. That is, there is a computable function } h \text{ and a computable sequence } (E_s) \text{ of finite sets of dyadic rational numbers such that } (E_s) \text{ is a computable } h\text{-}\text{enumeration of the left Dedekind cut } L_x \text{ of } x_A \). Suppose w.l.o.g. that, for any \( s \), if \( \sigma \in E_s \), then \( \tau \in E_s \) for any \( \tau \) such that \( l(\sigma) \leq l(\tau) \) and \( \tau \leq_L \sigma \). Let \( r_s \) be the maximal element of \( E_s \) and \( A_s := \{ n : r_s(n) = 1 \} \), i.e., \( r_s = x_{A_s} \) for any \( s \). Then, we have \( \lim_{s \to \infty} x_{A_s} = \lim_{s \to \infty} r_s = x_A \) and hence \( \lim_{s \to \infty} A_s = A \). Since \( (A_s) \) is obviously a computable sequence of finite subsets of natural numbers, it suffices to show that there is a computable function \( g \) such that \( (A_s) \) is a \( g\text{-}\text{enumeration of } A \). Now we define the computable function \( g : \mathbb{N} \to \mathbb{N} \) inductively as follows. For any \( n \in \mathbb{N} \) suppose that \( g(m) \) is defined for any \( m < n \). To define \( g(n) \), let’s estimate first how many times the membership of \( n \) to \( A_s \) can be changed for different \( s \) at all. Let \( \sigma \) be a binary word of the length \( n \). If there are two natural numbers \( s < t \) such that \( A_s \upharpoonright n = A_t \upharpoonright n = \sigma \) and \( n \notin A_s \) and \( n \in A_t \), then, we have \( r_s = x_{A_s} < \sigma 1 \) and \( \sigma 1 \leq x_{A_t} = r_t \). Here we regard the binary word \( \sigma 1 \) as a dyadic rational...
number. This implies that $\sigma 1 \notin E_s$ and $\sigma 1 \in E_t$ by the choice of the sequence $(E_s)$. Similarly, for the case $n \in A_s$ and $n \notin A_t$, we have $\sigma 1 \in E_s$ and $\sigma 1 \notin E_t$. Since $(E_s)$ is an $h$-enumeration of $I_2$, there are at most $h(\sigma 1)$ non-overlapping pairs $(s,t)$. This means that, for any $s < t$, if $n \in A_s \Delta A_t$ and $A_s \upharpoonright n = A_t \upharpoonright n = \sigma$, then $\sigma 1 \in E_s \Delta E_t$. Therefore, $A_s(n)$ can be changed at most $g(n)$ times where $g(n)$ is defined by

$$g(n) := \sum \{g(m) : m < n\} + \sum \{h(\sigma 1) : \sigma \in \{0,1\}^n\}.$$ 

Thus, $(A_s)$ is a computable $g$-enumeration of $A$ and hence $A$ is an $\omega$-c. e. set, because $g$ is obviously a computable function. That is, $x_A$ is $\omega$-binary computable.

**Corollary 4.4** The class $\omega\text{-}\text{dEC}$ is incomparable with $\text{WC}$ and hence is not closed under addition and subtraction.

## 5 Cauchy computability

In this section we discuss the Cauchy computability. We will show that, the general hierarchy theorem for Cauchy computability holds. However, the classes $k\text{-}\text{cEC}$ and $s\text{-}\text{cEC}$ are incomparable with the class $\text{SC}$, and the class $s\text{-}\text{cEC}$ is properly included in $\text{WC}$ but it is not closed under addition. In addition, we will show that the Cauchy computability is weaker than the binary computability on each level. Namely, the $\delta\text{-}\text{bEC} \subseteq \delta\text{-}\text{cEC}$ for $\delta \in \mathbb{N} \cup \{\ast, \omega\}$. Furthermore, the class of $\omega$-Cauchy computable reals is a closed field which contains properly the classes $\text{WC}$.

Let us begin with the hierarchy of Cauchy computable reals. We show a theorem similar to Theorem 2.2.

**Theorem 5.1** (Hierarchy Theorem) Let $f$, $g$ be computable functions. If there are infinitely many $n$ such that $f(n) < g(n)$, then there exists a $g$-$\text{cEC}$ real which is not $f$-$\text{cEC}$, i. e., $g$-$\text{cEC} \setminus f$-$\text{cEC} \neq \emptyset$.

**Proof.** Let $f, g : \mathbb{N} \rightarrow \mathbb{N}$ be two computable functions such that $f(n) < g(n)$ holds for infinitely many $n \in \mathbb{N}$. We are going to construct a computable sequence $(x_s)$ of rational numbers such that $(x_s)$ converges $g$-effectively to a real $x$ which is not $f$-Cauchy computable. That is, $x$ satisfies, for any $e \in \mathbb{N}$ the following requirements:

- $N (x_s)$ converges $g$-effectively to $x$,
- $R_e$ if $(\varphi_e(s))_s$ converges $f$-effectively, then $x \neq \lim_{s \rightarrow -\infty} \varphi_e(s)$,

where $(\varphi_e)$ is an effective enumeration of all computable partial functions $\varphi_e : \mathbb{N} \rightarrow \mathbb{Q}$.

The construction is a diagonalization with finite injury. The strategy to satisfy a single requirement $R_e$ is quite straightforward. Let $I_e := [a, b]$ be a rational interval with length $2^{-n_e+2}$ for some $n_e \in \mathbb{N}$ such that $f(n_e) < g(n_e)$. Divide it equally into four subintervals $I_i := [a_i, a_{i+1}]$, for $i < 4$, of the length $2^{-n_e}$. Let $b_1 := a_0 + 3 \cdot 2^{-(n_e+2)}$ and $b_2 := a_2 + 2^{-(n_e+2)}$. Notice that $2^{-n_e} \leq 2^{-n_e} + 2^{-n_e+1} = |b_1 - b_2| < 2^{-n_e+1}$. Define $x_s := b_1$ as long as the sequence $(\varphi_e(s))_s$ does not enter the interval $I_1$. Otherwise, if $\varphi_e(s)$ enters into the interval $I_1$ for some $s \geq n_e$, then let $x_s := b_2$. Later on, if $\varphi_e(t)$ enters $I_3$ for some $t > s$, then let $x_t := b_1$ again, and so on. If $(\varphi_e(s))_s$ converges $f$-effectively, then $(x_s)$ can be changed at most $f(n_e) + 1 \leq g(n_e)$ times. This guarantees that the sequence $(x_s)$ converges $g$-effectively and $\lim_{s \rightarrow -\infty} \varphi_e(s)$.

To satisfy all the requirements simultaneously, we use a finite injury priority construction. We change the above strategy a little bit as follows. For any requirement $R_e$, we choose a base interval $I_{e-1}$ with rational endpoints in which the strategy to satisfy $R_e$ is implemented. Of course, this interval can be changed finitely many times during the construction. Then we choose a natural number $n_e$ large enough such that $f(n_e) < g(n_e)$ and $2^{-n_e+1} \leq l(I_{e-1})$, where $l(I)$ is the length of the interval $I$. Now, we can choose four rational numbers $a_e, b_e, c_e, d_e \in I_{e-1}$ such that

$$(2) \quad a_e < b_e < c_e < d_e, \quad b_e - a_e = d_e - c_e = 2^{-(n_e+1)} \quad \text{and} \quad c_e - b_e = 2^{-n_e},$$

and let $I_e := [a_e; b_e]$ and $J_e := [c_e; d_e]$. The maximal and minimal distances between $I_e$ and $J_e$ are $2^{-n_e+1}$ and $2^{-n_e}$, respectively. Our task now is to determine, which interval, i. e., $I_e$ or $J_e$, should be chosen as the base.
interval of the requirement $R_{e+1}$. The criterion is simple. Namely, every real of this base interval should satisfy the requirement $R_e$. Thus, as default we can choose $I_e$. However, if for some $s \geq n_e$, $\varphi_e(s)$ enters $I_e$, then we change it to be $J_e$, or more simply, we exchange the intervals $I_e$ and $J_e$. If there is a $t > s$ such that $\varphi_e(t)$ enters the changed interval $I_e$, then we change it back, i.e., we exchange $I_e$ and $J_e$ again, and so on. At each stage $s$, we can define $x_s$ as the middle point of the current interval $I_e$. This means that, every time if we change the interval, the sequence $(x_s)$ will have a jump of the distance between $2^{-n_e}$ and $2^{-((n_e+1)}$. Therefore, the base interval of $R_e$ should be changed at most $f(n_e)$ times if the sequence $(\varphi_e(s))$, converges $f$-effectively and the corresponding sequence $(x_s)$ converges $g$-effectively.

**Corollary 5.2** For any $k \in \mathbb{N}$ we have $k\text{-cEC} \subseteq (k+1)-\text{cEC} \subseteq *\text{-cEC} \subseteq \omega\text{-cEC}$.

The next theorem shows that the classes $k\text{-cEC}$ are incomparable with $\text{SC}$ for all $k \geq 1$ and they are all contained properly in the class $\text{WC}$.

**Theorem 5.3**

1. $1\text{-cEC} \not\subseteq \text{SC}$.
2. $\text{LC} \not\subseteq *\text{-cEC}$.
3. $*\text{-cEC} \subseteq \text{WC}$.

**Proof.**

1. It is shown in [1] that if $A, B \subseteq \mathbb{N}$ are Turing incomparable c. e. sets, then the real $x_{A \oplus \overline{B}}$ is not semi-computable, where $A\oplus \overline{B} := \{2n : n \in A\} \cup \{2n+1 : n \in \overline{B}\}$ and $\overline{B}$ is the complement of set $B$. Let $(A_s)$ and $(B_s)$ be effective enumerations of two Turing incomparable c. e. sets $A$ and $B$, respectively, such that

$$A_0 = \emptyset \& (\forall s \in \mathbb{N}) (A_{2s} = A_{2s+1} \& |A_{2s+2} \setminus A_{2s+1}| = 1),$$

$$B_0 = \emptyset \& (\forall s \in \mathbb{N}) (B_{2s+1} = B_{2s+2} \& |B_{2s+1} \setminus B_{2s+2}| = 1).$$

This means that $|(A_{s+1} \oplus \overline{B}_{s+1}) \Delta (A_s \oplus \overline{B}_s)| = 1$ for all $s \in \mathbb{N}$, where $\Delta$ is the symmetrical difference of the sets defined by $C\Delta D := (C \setminus D) \cup (D \setminus C)$. Let $x_s := x_{A_s \oplus \overline{B}_s}$. Then $(x_s)$ is a computable sequence of rational numbers converging to $x_{A \oplus \overline{B}}$. Because $|x_{s+1} - x_s| = 2^n$ for $n \in (A_{s+1} \oplus \overline{B}_{s+1}) \Delta (A_s \oplus \overline{B}_s)$ and $n \neq m$ if $n \in (A_{s+1} \oplus \overline{B}_{s+1}) \Delta (A_s \oplus \overline{B}_s)$ and $m \in (A_{t+1} \oplus \overline{B}_{t+1}) \Delta (A_t \oplus \overline{B}_t)$ for different $s$ and $t$. The sequence $(x_s)$ converges also 1-effectively. That is, $x_{A \oplus \overline{B}}$ is an 1-cEC but not semi-computable reals.

2. We construct an increasing computable sequence $(x_s)$ of rational numbers such that the limit $x := \lim_s x_s$ is not $k\text{-cEC}$ for any $k \in \mathbb{N}$. That is, $x$ satisfies for all $i, j \in \mathbb{N}$ the following requirements $R_{(i,j)}$ if $(\varphi_i(s))$ converges $j$-effectively, then $x \neq \lim_s \varphi_i(s)$,

where $(\varphi_i)$ is an effective enumeration of all computable partial functions $\varphi_i : \mathbb{N} \rightarrow \mathbb{Q}$. To satisfy a single requirement $R_{e(i,j)}$ for $e = (i,j)$, we choose a rational interval $[a,b]$. Let $n$ be the minimal natural number such that $2^n \geq 3(j+1)(b-a)$. Define $a_i := a + i \cdot 2^{-n}$ for $i \leq 3(j+1)$ and $a_{3(j+1)+1} = b$. Then the intervals $I_t := [a_t : a_{t+1}]$ have the length $2^{-n}$ for any $i < 3(j+1)$. We define $x_0$ as the middle point of the interval $I_1$. If $\varphi_i(s)$ enters $I_1$ for some $s \geq n$, then define $x_s$ as the middle point of the interval $I_1$. If there is a $t > s$ such that $\varphi_i(t)$ enters $I_t$, then define $x_t$ as the middle point of the interval $I_t$, and so on. In general, if $x_{s_j} \in I_{3k+1}$ and $\varphi_i(s_j) \in I_{3k+1}$ for some $s_j \geq s_1$, then redefine $x_{s_j}$ as the middle point of $I_{3k+1}$. If $(\varphi_i(s))$ converges $j$-effectively, then we can always find a correct $x$ which differs from the limit $\lim_s \varphi_i(s)$, because $\varphi_i(1) \in I_{3k+1}$ and $\varphi_i(2) \in I_{3k+1}$ implies that $2^{-n+1} \leq |\varphi_i(s_1) - \varphi_i(s_2)| \leq 2^{-n+2}$.

To satisfy all requirements, it succeeds to apply the strategy to an interval tree and use the finite injury priority construction. We omit the details here.

3. Let $(x_s)$ be a computable sequence of rational numbers which converges $k$-effectively to some $k$-computable real $x$. For any $n \in \mathbb{N}$ let $S_n := \{s \in \mathbb{N} : 2^{-n} \leq |x_s - x_{s+1}| < 2^{-n+1}\}$ Then, we have the following estimation:

$$\sum_{s \in \mathbb{N}} |x_s - x_{s+1}| = \sum_{n \in \mathbb{N}} \sum_{s \in S_n} |x_s - x_{s+1}|$$

$$= \sum_{n \in \mathbb{N}} \sum_{s \in S_n, k \leq s} |x_s - x_{s+1}| + \sum_{s \in S_n, k > s} |x_s - x_{s+1}|$$

$$\leq \sum_{n \in \mathbb{N}} (n \cdot 2^{-n+1} + k \cdot 2^{-n+1}) \leq 8 + 2k.$$
That is, \( x \) is a weakly computable real. Therefore, \( \ast\text{-}EC \subseteq WC \). By 2., there is a left computable real which is not \( \ast\text{-}cEC \). Therefore the inclusion is also proper.

The next theorem shows that \( \ast\text{-}cEC \) is not closed under addition and subtraction.

**Theorem 5.4** There are \( x, y \in 1\text{-}cEC \) such that \( x - y \notin \ast\text{-}cEC \). Therefore, \( k\text{-}cEC \) and \( \ast\text{-}cEC \) are not closed under addition and subtraction for any \( k > 0 \).

**Proof.** We will construct two computable increasing sequences \( (x_s) \) and \( (y_s) \) of rational numbers which converge 1-effectively to \( x \) and \( y \), respectively, while their difference \( z := x - y \) is not \( \ast\text{-}cEC \). That is, \( z \) is not \( k\text{-}Cauchy \) computable for any constant \( k \) and hence \( z \) has to satisfy, for all \( i, j \in \mathbb{N} \), the following requirements:

\[
R_{(i,j)} \quad \text{if } (\varphi_i(s))_s \text{ converges } j\text{-effectively, then } \lim_{s \to \infty} \varphi_i(s) \neq z,
\]

where \( (\varphi_i) \) is an effective enumeration of all partial computable functions \( \varphi_i : \subseteq \mathbb{N} \to \mathbb{Q} \).

To satisfy a single requirement \( R_e \) for \( e := (i, j) \), we choose two natural numbers \( n_e \) and \( m_e \) large enough such that \( n_e < m_e \) and \( m_e - n_e \geq j + 3 \). As default, let \( x_s = y_s = z_s = 0 \) as long as no \( t \leq s \) is found such that \( |z_s - \varphi_i(t)| < 2^{-(m_e+2)} \) holds. However, if there exists a \( t_0 \geq m_e \) at some stage \( s_0 \) such that \( |z_s - \varphi_i(t_0)| < 2^{-(m_e+2)} \) holds, then we define \( z_{s_0} := x_{s_0} - y_{s_0} \) for

\[
|x_{s_0} - \varphi_i(t_0)| \geq |z_{s_0} - z_s| - |z_s - \varphi_i(t_0)| > 3 \cdot 2^{-m_e+2}.
\]

If at a later stage \( s_1 > s_0 \), there exists a \( t_1 > t_0 \) such that \( |z_{s_0} - \varphi_i(t_1)| < 2^{-m_e+2} \) holds, then we define

\[
x_{s_1} := x_{s_0} + 2^{-(n_e+2)} \quad \text{and} \quad y_{s_1} := y_{s_0} + 2^{-(n_e+2)} + 3 \cdot 2^{-m_e+1},
\]

and so on. Notice that we use \( n_e + 2 \) in (4) in stead of \( n_e + 1 \) in (3). But similarly, we have the inequality

\[
|z_{s_1} - \varphi_i(t_1)| > 2^{-m_e}.
\]

Now we are going to discuss the \( \omega\text{-}Cauchy \) computability. According to Definition 2.4, the Cauchy computability counts the number of \( n \)-jumps whose distances are between \( 2^{-n} \) and \( 2^{-(n+1)} \). In [13] the authors together with Gengler and von Braunmühl have explored a similar notion called divergence bounded computability where the jumps of distance larger than \( 2^{-n} \) are counted. A real \( x \) is called divergence bounded computable (dbc, for short) if there exist a computable function \( h \) and a computable sequence \( (x_i) \) of rational numbers which converges to \( x \) such that, for any \( n \in \mathbb{N} \) there are at most \( h(n) \) non-overlapping pairs \( (i, j) \) of indices with \( |x_i - x_j| \geq 2^{-n} \). Obviously, for divergence bounded computability, a corresponding \( k\text{-}computability \) can not be introduced. However, it is not difficult to see that a real is dcb iff it is \( \omega\text{-}Cauchy \) computable. It is showed in [13] that the class of all dcb reals is a closed field and is strictly between the classes \( WC \) and \( CA \). Therefore, we have the following theorem straightforwardly.

**Theorem 5.5** The class \( \omega\text{-}cEC \) is a field and \( WC \not\subseteq \omega\text{-}cEC \subseteq CA \).

At last, we compare the Cauchy computability with binary and Dedekind computability. Let’s look at the \( \omega\)-computability first. By Theorem 4.3, the \( \omega\)-binary computability is equivalent to the \( \omega\)-Dedekind computability. The next theorem shows that they are both stronger than \( \omega\text{-}Cauchy \) computability.

**Theorem 5.6** \( \omega\text{-}bEC = \omega\text{-}dEC \not\subseteq \omega\text{-}cEC \).
Proof. By Theorem 3.4 and Theorem 5.5, it suffices to prove the inclusion \( \omega \cdot b \text{EC} \subseteq \omega \cdot c \text{EC} \). Suppose that \( x \in \omega \cdot b \text{EC} \), i.e., \( x = x_A \) for an \( \omega \)-c.e. set \( A \). Let \( h \) be a computable function and \( (A_t) \) be a computable \( h \)-enumeration of \( A \). Notice that, if \( A_s \upharpoonright n = A_t \upharpoonright n \), then \( |x_{A_s} - x_{A_t}| \leq 2^{-n} \) for any \( s, t \) and \( n \). This means that, the computable sequence \( (x_s) \) defined by \( x_s := x_{A_s} \) for all \( s \) converges to \( x_A \) \( g \)-effectively, where \( g \) is a computable function defined by \( g(n) := \sum_{i \leq n} h(i) \). Thus, \( x_A \) is \( \omega \cdot c \text{EC} \).

For a relationship between Dedekind and Cauchy computability in the lower levels, the following theorem follows immediately from Lemma 4.1 and Theorem 5.3.

**Theorem 5.7** \( 0 \cdot \text{dEC} = 0 \cdot \text{cEC} \) if, and for any \( n \geq 1 \) or \( n = * \), \( n \cdot \text{dEC} \not\subseteq n \cdot \text{cEC} \) and \( n \cdot \text{cEC} \not\subseteq n \cdot \text{dEC} \).

However, the next theorem shows that the Cauchy computability is strictly weaker than binary computability in general.

**Lemma 5.8** \( 0 \cdot \text{bEC} = 0 \cdot \text{cEC} \) and \( n \cdot \text{bEC} \subseteq k \cdot \text{cEC} \) for any \( k \geq 1 \) or \( k = * \).

Proof. For the inclusion part, let \( x_A \) be a \( k \)-binary computable real. That is, \( A \) is a \( k \)-c.e. set and \( (A_s) \) a computable \( k \)-enumeration of \( A \), then \( (x_{A_s}) \) is a computable sequence of rational numbers which converges to \( x_A \) \( k \)-effectively and hence \( x_A \) is \( k \)-Cauchy computable.

For the inequality, it suffices to prove the relation \( 1 \cdot \text{cEC} \not\subseteq * \cdot \text{bEC} \).

We will construct a computable sequence \( (x_s) \) of rational numbers which converges \( 1 \)-effectively to a non-*cEC* real \( x_A \), i.e., \( A \) is not \( k \)-c.e. for any constant \( k \). Then the set \( A \) has to satisfy for all \( i, j \) the following requirements:

\[
R_{(i,j)} \quad \text{if } (W_{i,s})_{s \in \mathbb{N}} \text{ is a } j \text{-enumeration, then } \lim_{s \to \infty} W_{i,s} \neq A,
\]

where \( (W_e) \) is a computable enumeration of all c.e. subsets of \( \mathbb{N} \) and \( (W_e,s) \) is its uniformly computable approximation. The strategy to satisfy a single requirement \( R_e \) for \( e = (i, j) \) is as follows. We choose an interval \( I_e = [n_e, m_e] \) of natural numbers such that \( m_e - n_e > 2j \). This interval is preserved exclusively for the requirement \( R_e \). At the beginning, let \( x_0 := 2^{n_e} \) (\( n_e \) is put into \( A \)). If at some stage \( s_0 \), \( n_e \) enters \( W_{i,s_0} \), then define \( x_{s_0+1} := x_{s_0} - 2^{-m_e} \) (\( n_e \) leaves \( A \)) and let \( m_e := m_e - 1 \). If at a later stage \( s_1 > s_0 \), \( n_e \) leaves \( W_{i} \), then define \( x_{s_1+1} := x_{s_1} + 2^{-m_e} \) (\( n_e \) enters \( A \)) and let \( m_e := m_e + 1 \), and so on. We take this action at most \( j \) times. Thus, if \( (W_{i,s})_{s \in \mathbb{N}} \) is a \( j \)-enumeration, then \( R_e \) will be satisfied eventually. The sequence \( (x_s) \) defined in this way converges obviously \( 1 \)-effectively.

If we choose the sequence \( (I_e) \) of intervals properly, the above strategy can be applied simultaneously to satisfy all requirements. That is, there is an \( 1 \)-cEC real which is not \( * \cdot \text{bEC} \).

### References


